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OCEANIC TURBULENCE
(REVIEW)

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Principal Investigator
Takashi Ichiye

PREFACE

This review is a kind of harvest of my random study and reading on oceanography for many years, although actual enterprise of writing began just this past summer. When I started some experimental work on turbulent diffusion in the sea, I felt an urgent need for a quick reference to the studies on turbulence in the ocean. However, there is neither published nor unpublished literature available which might answer such a need, although meteorology has already a series of monographs and books on atmospheric turbulence by Lettau (1939), Sutton (1948), Ellison (1956), Priestley (1959) and Pasquill (1962). Of course there is some similarity between oceanic and atmospheric turbulence, but quite a number of processes including those at the free surface and at lateral boundaries are peculiar to oceanic turbulence.

Then, I had a chance to work at the Weather Bureau in Washington, D.C. this summer through a project on storm surges. Mr. D. Lee Harris, the head of the branch for this project, suggested that I write a review on vertical eddy viscosity, because determination of this parameter is important to numerical prediction of storm surges in shallow water and yet there is quite a lot of controversy among the studies done before. A draft of the first five chapters was thus finished during my stay in Washington, even though the life in this city was filled with so

many joys that I could hardly concentrate on writing the draft.

After coming back to Tallahassee, I decided to add a few chapters, in order to satisfy my need for reference to the papers related to oceanic turbulence in general and to clear up, once and for all, my desk which was always jammed with so many books and journals on turbulence that I could hardly find some references quickly. The additional parts, then, expanded into more than twice the volume of the initial draft, reflecting the situation that turbulence which was rather a stepchild in oceanography, is related to its many phases in some way or other. How about my desks? More crowded than before, indicating that this review is far from complete.

At first I tried to be consistent in style, mathematical notations, etc. throughout the whole chapters. Soon I found that this task was as hard as keeping my desks clear. So many of the mathematical notations are presented as the original authors used them, although in some cases they are corrected in order to avoid confusion at least in each chapter. Some inconsistencies in mathematical notations should be blamed mostly on my carelessness and negligence, but partly due to the different usages by many authors reviewed. Such heterogeneity might be inevitable to the present developing stage of researches on oceanic turbulence.

The spectrum of nationalities of the papers reviewed reflects neither quality nor quantities of works done about oceanic turbulence at corresponding countries. The Russian papers are reviewed rather abundantly, because they were not well known in this country and also because I am struggling to master this

difficult language for the third time in my life. The Japanese papers are quoted quite often because they are not well known here and because I happen to be familiar with this language. I hope that the most frequent quotations from my own papers would be justified on the same reason as mentioned above, i.e., these papers are deplorably unread.

I am greatly indebted to the Office of Naval Research through its support of the research project "Turbulent Diffusion and Surface Phenomena in the Ocean". Also, I am very grateful to Mr. Harris and his group at the Weather Bureau for their hospitality during my stay there. Last, but not least, I thank Mrs. Susy Bowman who typed the ditto originals and deciphered the hardly legible manuscript.

Takashi Ichiye

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TURBULENCE IN THE OCEAN
(Review)

Takashi Ichiye

I. Introduction

In the preface of the book "The Structure of Turbulent Shear Flow" A.A. Townsend wrote that experimental knowledge on turbulent shear flows was being accumulated too rapidly to produce new theories. Unfortunately this is not true for oceanic turbulence. There have been very few experimental studies on structure of turbulent flows in the ocean, partly owing to difficulties in instrumentation which must be operated under a rather unfriendly environment full of corrosive medium, uncontrollable living things and violent forces such as waves and winds. However, most of the blames might be placed on a lack of interests of physical oceanographers. Descriptive oceanographers do not want to recognize importance in turbulence in order to keep a sacred cow of oceanography, geostrophic model clean. Theoretical oceanographers also try to minimize the influence of turbulence in order to keep their mathematics straightforward. At the most they reluctantly include in their equations terms containing eddy viscosities, the values of which may be determined at their convenience!

However, some fluid dynamicists studying turbulence, on one hand, have been more and more interested in turbulence in the ocean. One of reasons for such is that oceans provide an ideal natural laboratory for turbulence of a wide range of scales, clearly defined by boundaries of surface and bottom and coasts. (Frenkiel, 1962) On the other hand, some

engineers and physicists, engaging in study of underwater acoustics or pollution problems are also increasing their interests in oceanic turbulence. Therefore, it will be expected that in some future every oceanographer should talk about turbulence, at least to conceal his ignorance on dynamics of the ocean flows.

Why oceanographers should have a knowledge of turbulence? Although most of movements of water in the ocean is slow, it becomes turbulent owing to a large scale of motion in the ocean. However, general problems in physical oceanography are particularly those of descriptive oceanography are to obtain information on movement of the ocean water in the averaged condition over a certain period of time and/or a stretch of space. Classical instruments for measuring oceanographic data like propeller type current meters and the data procession such as dynamic calculation of ocean currents from hydrographic data were intended, though without any special attention, to obtain data averaged over time and/or space, rather than to obtain instantaneous values. Even so, it is very important to have an idea how representative are such data as obtained by classical means, what kind of average should be used, etc. In order to answer properly such questions basic knowledge on turbulence is essential.[*]

2. Mixing length theory

Generally speaking turbulence problems become important to oceanography or meteorology through relationships between turbulence and mean motion of the water or atmosphere. In order to obtain such relationships, hydrodynamic equations of motion are averaged over a certain period of time, a domain in space or a great number of realizations of fluid motion. The effect of turbulence on the mean flow is expressed by terms called Reynolds stresses.

The most important effect of turbulence on the mean flow is to transport momentum, heat and properties of fluid in a different way from laminar flow. As the first step to incorporating such effect mathematically into the Navier-Stokes equations, an analogy with molecular diffusion was assumed. The mean flow is treated like a "laminar flow" using eddy viscosity. However, the eddy viscosity has an order of magnitude of 100 to 10^{10} times of molecular viscosity, depending on a scale of motion considered. Moreover, the eddy viscosity is variable not with space and time, but also with stability and other dynamic characteristics of the mean flow. Most of the efforts on study of turbulence in oceanography and meteorology before the recent decade were devoted to determing such dependency of eddy viscosity on various characteristics of the mean flow.

One such effort, which turned out to be quite useful to practical problems is a mixing length theory of Prantdl. Apparent success of this hypothesis in explaining many types of flows, both of laboratory scale and geophysical scale, lead many people

to accept it as an established fact. However, merits of this hypothesis are to give a heuristic explanation on problems such as vertical distributions of velocity of mean flow in the ocean and the atmosphere near the boundary. So far as we understand the limit of validity of the theory, it is quite useful to problems like estimates of wind stresses or evaporation near the surface from synoptic weather charts. Therefore, a historical interest, it is worthwhile to practical as well as historical interests to describe the development of the theory and its application to oceanographic problems. In fact, many Russian works on oceanic turbulence were done in recent years to a new development of mixing length theory, with a certain amount of success in heuristic approach to the vertical mixing and air-sea interchange problems.

Historically speaking, the idea of mixing length was already introduced by Bousinesq at the end of last century but Prandtl (1925) elaborated it later (Dryden, Murnagan and Bateman 1932). The application of the mixing length theory to oceanographic and meteororlogical problems was initiated by Prandtl (1932) and by Rossby (1932). After then, many elaborations, developments, misuse and abuse of the theory followed. Development by Rossby and Montgomery (1935, 1936) contributed very much to spreading the influence of the theory among meteorologists and oceanographers. Subsequent applications of the theory were extended to determination of vertical profiles of wind or ocean currents near the boundary, estimation of wind stresses over the ocean and ground (Wilson, 1960), heat transport and evaporation (Priestley, 1959). Wind stress formula and evaporation formula which are still used to

compute wind stresses and evaporation rate from synopitic data are mostly derived from the mixing length theory. Among them we can name the evaporation formula derived by Montgomery (1940), Sverdrup (1937) and Thornthwaite (1939). The theory also was applied to a problem of shallow water currents produced by winds, first by Rossby and Montgomery (1935) and then by Hellstrom (1941) in his study of storm surge.

Essentially, the mixing length theory (or more properly it might be called hypothesis, is one of the assumptions the Reynolds stresses in terms of quantities represented the structure of the mean flow. With conventional coordinate system in meteorology and oceanography, which x and y axis are taken horizontally and z-axis is taken vertically upwards, the mixing length hypothesis in most essential form is expressed by an equation.

$$\overline{u'w'} = \overline{w'l'}\,(\partial \bar{u}/\partial z) \qquad (2.1)$$

$$-\overline{w'l'} = l^2\,|\partial \bar{u}/\partial z| \qquad (2.2)$$

in which u' and w' are turbulent velocity components in x and z directions. $\bar{u}$ is the mean flow. The quantities l' and l are called mixing length. The quantity l' is defined in analogy with a mean free path of gas molecules in dynamical theory of gases. The relation similar to equation (1) was already derived by Taylor (1915) and Schmidt (1917). Prandtl's contribution is the derivation of the second relation which defines l as a mixing length in Prandtl's sense.

3. Planetary Boundary Layers

Rossby and Montgomery (1935) proposed a hypothesis combining earlier theory of Rossby (1932) on application of a mixing length hypothesis to the Ekman layer in the atmosphere and the ocean and the theory of Prandtl (1932) on a frictional boundary layer. According to them, the frictional layer between the ground or the sea surface and the layer where geostrophic flows are valid is divided into two parts: a frictional boundary layer, in which pressure gradient is balanced by shearing stresses and a planetary boundary layer in which Coriolis' force has the same importance as these two forces. The depth of frictional boundary layer and planetary boundary layer are indicated by H and h, respectively.

In the frictional boundary layer, the relation between mixing length (ℓ), shearing stress (τ) and height from the boundary (z) is from Prandtl's length hypothesis given by

$$\tau = -\rho\,\overline{u'w'} = \rho\,\ell^2 \left(\frac{dU}{dz}\right)^2 \qquad (3\cdot3)$$

$$\ell = k_0\,(z+z_0) \qquad (3\cdot4)$$

where z is measured above the mean level of roughness elements of the boundary, z_0 is the roughness parameter which is almost equal to 1/30th of the average height ϵ of roughness elements and dU/dz represents the shear of the mean motion. The stresses too are assumed to be constant with height in the frictional boundary layer. The well known logithmic profile (5) of the mean flow U is obtained by integrating (3) with z.

$$U = (1/k_0)\sqrt{\tau_0/\rho}\;\log[(z+z_0)/z_0] \qquad (3\cdot5)$$

Although equation (3·5) is derived for the air flow near the ground, the equation similar is considered to give the profiles of the wind over the water or of the drift current below the sea level. For the drift current, the values of z_0 should be modified by taking into account of surface waves.

In the planetary boundary layer, Rossby and Montgomery assumed that the rate of shear is constant and that the mixing length decreases linearly with height and vanishes at the top of the layer. When the mean velocity is expressed by complex form like $W = U + iV$ these assumptions lead to the equations

$$dW/dz = Ke^{i\psi} \qquad (3\cdot6)$$

$$l = k(h-z)/\sqrt{2} \qquad (3\cdot7)$$

where K and k are constants and ψ is variable with z. The origin of z is taken at the top of the frictional boundary layer. The equation of motion becomes:

$$W = U_g - \frac{i}{f}\frac{d}{dz}\left(A\frac{dW}{dz}\right) \qquad (3\cdot8)$$

in which U_g is geostrophic current and A is given by Kl^2. The relationship between ψ and z can be obtained by differentiating (3·8) about z. Thus, we have:

$$z = h\{1 - \exp(\psi - \psi_s)\sqrt{2}\} \qquad (3\cdot9)$$

where ψ_s is the angle between the wind at the top of frictional boundary layer $z=0$. There are the following relationships among various constants.

$$K = \sqrt{2}f/3k^2 \qquad (3\cdot10)$$

$$h = (9/2)k^2(U_g/f)\sin\psi_s \qquad (3\cdot11)$$

$$\tau_0 = \rho f^2 h^2/9k^2 \qquad (3\cdot12)$$

The wind profiles in the planetary boundary layer are expressed by:

$$W/U_g = 1 + \sqrt{\tfrac{3}{2}}\sin\psi_s \cdot \exp\left[i(\pi - \beta + \psi_s) + \left(i + \tfrac{1}{\sqrt{2}}\right)(\psi - \psi_s)\right] \qquad (3\cdot13)$$

where:

$$\tan\beta = \sqrt{2}$$

The eddy-viscosity is expressed from Prandtl's hypothesis by

$$\eta = \rho l^2 K = \{\rho f/(3\sqrt{2})\}(h-z)^2 \qquad (3\cdot14)$$

The condition that mixing length is continuous at the boundary between planetary and frictional boundary layer leads to

$$k_0(H+Z_0)=k\,h/\sqrt{2} \quad \text{or} \quad H=0.12h \qquad (3\cdot15),(3\cdot16)$$

Also, the conditions that wind velocity and direction and frictional drag are continuous there lead to the relationships between ψ_s and U_g such as:

$$N=\frac{U_g}{fZ_0}=\frac{2\sqrt{2}\,k_0}{9k^3}\,\frac{1}{\sin\psi_s}\,e^{\frac{2k_0}{3k}\left(\cot\psi_s-\frac{1}{\sqrt{2}}\right)} \qquad (3\cdot17)$$

From equation 17, ψ_s can be determined when U_g and Z_0 are given. Then h can be determined from equation 11, using this value of ψ_s. The total height of the layer of frictional influence is thus given by:

$$H+h=1.12h=\frac{5.04k^2}{f}\,U_g\sin\psi_s \qquad (3\cdot18)$$

A vertical distribution of eddy viscosity obtained from this theory is shown in Fig. 1, by taking $Z_0=32$ cm, $f=10^{-4}$ sec^{-1} and $U_g=7.5$ and 20 mps, respectively. The dotted curves indicate the distributions in which eddy viscosity in the layer above the planetary boundary layer is taken to be equal to 50 $\frac{cm^2}{sec}$ instead of zero, according to the estimation of Richardson (1926).

Recent development on the planetary boundary layer based on mixing length theory was reviewed by Ellison (1956) and Blackadar (1962). The latter, especially, introduced a continuous distribution of the mixing length like

$$l=k_0z\,(1+k_0z/l_\infty)^{-1} \qquad (3\cdot19)$$

In this distribution, l increases as k_0z near to the ground but it approaches to the constant value l_∞ in higher levels.

Wind-driven currents as well as wind near the ground were first discussed mathematically by Ekman (190), who introduced the idea of eddy viscosity, although he assumed the constant value

of it in order to avoid mathematical complexity. Vertical distributions of flows in the friction layer of the ocean and the atmosphere predicted by his theory have not been changed substantially in later developments by his successors. The hodographs of such flows called "Ekman's spiral" are referred almost in every text book of oceanography and meteorology. Rossby and Montgomery (1935) also applied their theory of the atmospheric boundary layer flow to wind-driven current in the upper layer of the ocean. Only the difference between the atmospheric and the oceanic boundary layer is in the surface roughness at the atmospheric boundary solid ground and at the oceanic boundary flexible sea surface.

It is well understood that the roughness of the ocean surface must be changed according to the heights of the surface waves, although mathematical formulation of such relationships might be quite intricate. Rossby and Montgomery (1935) simply assumed that the roughness parameter of the sea surface is proportional to the surface wave heights. Further, they speculated that the proportional coefficient is of an order of one, for the drift current and is about equal to 1/30for the wind over the sea surface. It seems to be more probable to consider that the roughness of the sea surface to the wind might be mainly due to the capillary waves of higher steepness superposed on larger waves, as suggested by Neumann (1952). Effects of waves on the drift currents or more generally the interaction of waves and currents is more intricate problem than speculated by Rossby and Montgomery. The mathematical treatment of hydrodynamics

pertinent to this problem is now undertaken by Miles (1960), Longuet-Higgins and Stewart (1961) and others. The discussion of this problem will need quite a space.

The depth of the planetary boundary layer h of the ocean, according to Rossby and Montgomery, from an argument similar to the one on the atmospheric boundary layer is given by

$$h = 3\frac{k}{f}\sqrt{\frac{\tau_o}{\rho_w}} = 3\frac{\gamma k}{f}\sqrt{\frac{\rho_a}{\rho_w}}\;W_o \qquad (3\cdot 20)$$

where ρ_a and ρ_w are the density of air and water, respectively. The wind stress τ_o and surface wind speed W_o are related by

$$\tau_o = \gamma\rho_a\;W_o^2 \qquad (3\cdot 21)$$

in which γ is a drag coefficient. The determination of this coefficient is a matter of controversy which was reviewed recently by Wilson (1960). Its value is dependent, not only on the roughness of the surface, but also on some specified height at which the wind speed is measured.

The angle φ_o between the wind stress and surface drift is given by:

$$\tan\varphi_o = 2\left(\sqrt{2} + 3Z\right)^{-1} \qquad (3\cdot 22)$$

in which:

$$Z = \frac{k}{k_o}\ln\left(\frac{3k^2}{k_o z_{ow} f}\sqrt{\frac{\tau_o}{2\rho_w}}\right) \qquad (3\cdot 23)$$

where z_{ow} is the roughness parameter of sea surface for the currents below. The wind factor, ratio of the surface drift to the wind speed is given by:

$$w_o/W_o = \frac{\gamma}{k_o}\left[\frac{\rho_a}{\rho_w}\left(\frac{2}{3} + \frac{2\sqrt{2}}{3}Z + Z^2\right)\right]^{\frac{1}{2}} \qquad (3\cdot 24)$$

These two quantities, angle between wind and surface drift and the wind factor were widely used to verify a theory, particularly of Ekman (1905), on drift currents because they are considered as the quantities determined easily, for instance,

from ships' logs. However, in order to test by observations the theory of drift currents either by Ekman or by Rossby and Montgomery, the following conditions on which the theory was based should be satisfied. These conditions are: (1) Deep sea without noticeable effect of bottom friction, (2) Open sea without effects of coastal boundaries, (3) Homogeneous density of water, (4) No surface slope and, (5) Wind in steady state. It is rather surprising that the angles between wind and surface drift were determined mostly in the shallow seas and land-locked such as the Baltic Sea (Dinklage, 1888; Witting, 1909), the Gulf of Bothnia (Palmen, 1931) and the Eastern Mediterranean (Forch; 1911). Only the data used by Galle (1910) were obtained in the equatorial regions of the Indian Ocean, indicating agreement of observed angles with those predicted by the Ekman theory. However, such agreement is not conclusive, because the North and the South Equatorial Currents are not only driven by wind, but also caused by the slope of sea surface. (See Defant(1961),pp412-418. The references are listed there.)

Apparently one of the most suitable places to test the Ekman's theory might be the Sargasso Sea or the counterpart in the Pacific Ocean. So far there is no observation for this purpose in such region exmpt the one by Stommel (1954), who tried to measure the drift currents in the Sargasso sea by use of drifting buoys, but without any conclusive result.

4. Application of mixing length theory to shallow water.

Rossby and Montgomery (1935) also applied the mixing length theory to shallow water currents for the first time. Instead of discussing general features of turbulence in the shallow

waters, they tried to explain the vertical distributions of currents measured within eight hours on a certain day at Buzzard Bay, Massachusetts, which is 17 m deep. Basic assumptions of their theory are that the mixing length is constant with depth and that the shearing stress decreases with depth in case wind blows against the tidal currents and it remains constant with depth in case wind blows with the currents.

In general equation of motion of tidal currents in a shallow sea is given by:

$$\frac{\partial u}{\partial t} = -g\frac{\partial \zeta}{\partial x} + \frac{1}{\rho}\frac{\partial \tau}{\partial z} \qquad (4\cdot 1)$$

On that specific day when the current was measured, the wind blew almost parallel to the long axis of the bay. In the morning, tidal currents flowed against the wind. The data showed that current speed was almost unchanged during the morning. In equation (4.1), $\partial u/\partial t$ is assumed to be vanish and the equation for τ_x between the surface and the depth of maximum speed is given by:

$$\tau_x/\rho = (\tau_0/\rho)\cdot(z/h) = l^2\,(\partial u/\partial z)^2 \qquad (4\cdot 2)$$

where τ_0 is the surface stress, h is the depth of the maximum velocity and the origin of z is taken at the layer of the maximum speed. With an assumption of constant mixing length, the vertical distribution of velocity and eddy viscosity are given by:

$$u_{max} - u = \frac{2h}{3l}\left(\frac{\tau_0}{\rho}\right)^{\frac{1}{2}}\left(\frac{z}{h}\right)^{3/2} \qquad (4\cdot 3)$$

$$\eta = \rho l\left(\frac{\tau_0}{\rho}\frac{z}{h}\right)^{1/2} \qquad (4\cdot 4)$$

By comparing (4.3) with observed velocity profiles, they obtained the ratio of mixing length l to total depth d as 0.051.

In the afternoon, the tidal current flowed in the same direction as the wind. The observed current showed linear decrease in speed with depth, leading to the assumption that the stress is constant with depth. Then, $\partial u/\partial t$ and $-g\,\partial\zeta/\partial t$ should balance each other. The constant value of τ_x is given by the last term of equation (4d). The mixing length can be determined since τ_x is equal to the surface wind stress τ_0 and $\partial u/\partial z$ is known from the observed current profile, thus obtained. The value of l is 0.052d, and is surprisingly close to the value estimated by use of the data for the current flowing against the wind.

An approach essentially similar to Rossby and Montgomery's study was taken by Kivisild (1954) in order to derive vertical distributions of velocity in a shallow water as one of problems pertinent to storm surges. Ichiye (1952a) also determined vertical distributions of velocity caused by a stationary wind blowing perpendicular to the coast under more general boundary conditions at the bottom than Kivisild.

Both studies are based on the equation of motion in the x-direction under steady condition

$$\partial \tau_x / \partial z = \rho g (\partial \zeta / \partial x) = i_x \qquad (4\cdot 5)$$

in which τ_x is the shearing stress. Inertia terms and Coriolis' term are neglected. Since i_x is constant with depth, integration of $(4\cdot 5)$ with z yields:

$$\tau_x - \tau_b = i_x z \qquad (4\cdot 6)$$

in which τ_b is the bottom stress, which must be dependent on roughness of the bottom and current close to the bottom.

The velocity profiles are also dependent on the boundary conditions at the bottom. Ichiye (1952a) considered three conditons at the bottom: (1) Bottom stress and bottom current are reverse to the surface stress, (2) Bottom stress is reversed to the surface stress and the bottom current and (3) the bottom current is reverse to the surface stress and the bottom stress. In the last two conditions, the stress and the current at the bottom are in the opposite direction. This is possible since the stress acts in the same direction as the strain or vertical gradient of velocity which is not always in the same direction as the velocity itself. The velocity distributions corresponding to three conditions are given by:

(first case) $U = \frac{2}{3}(U_*/\ell_h)(1+n)^{-1}\left[\,|(1+n)(Z/h)-n|^{\frac{3}{2}} - n^{\frac{3}{2}}\right] - U_* k_0^{-\frac{1}{2}}$ (4·7a)

(second case) $U = \frac{2}{3}(U_*/\ell_h)(1+n)^{-1}\left[\,|(1+n)(Z/h)-n|^{\frac{3}{2}} - n^{\frac{3}{2}}\right] + U_* k_0^{-\frac{1}{2}}$ (4·7b)

(third case) $U = \frac{2}{3}(U_*/\ell_h)(1-n)^{-1}\left[\,|(1-n)(Z/h)-n|^{\frac{3}{2}} - n^{\frac{3}{2}}\right] - U_* k_0^{-\frac{1}{2}}$ (4·7c)

respectively, in which k_0 is Karman's constant, n is equal to τ_0/τ_b, ratio of surface stress τ_0 to the bottom stress τ_b, ℓ_h is a ratio of mixing length ℓ to the total depth h and U_* is the friction velocity defined by

$$U_* = (\tau_0/\rho)^{\frac{1}{2}} \qquad (4\cdot 8)$$

This ratio n can be determined from the following relations for each case as a function of ℓ_h.

(first case) $C\left(1 - \frac{5}{2}n^{\frac{3}{2}} - \frac{3}{2}n^{\frac{5}{2}}\right) - n^{\frac{1}{2}}(1+n)^2 = 0$ (4·9a)

(second case) $C\left(1 - \frac{5}{2}n^{\frac{3}{2}} - \frac{3}{2}n^{\frac{5}{2}}\right) + n^{\frac{1}{2}}(1+n)^2 = 0$ (4·9b)

(third case) $C\left(1 - \frac{5}{2}n^{\frac{3}{2}} + \frac{3}{2}n^{\frac{5}{2}}\right) - n^{\frac{1}{2}}(1-n)^2 = 0$ (4·9c)

in which $C = 4k_0^{\frac{1}{2}}(15\ell_h)^{-1}$. These equations are obtained with the conditions that the mixing length is constant with depth and that the total mass transport vanishes owing to the assumption

that the wind blows perpendicular to the coast. The vertical distributions of velocity computed from these equations are given from these equations are given in Figure 2.

Kivisild's theory treated only the condition corresponding to the first case. In his discussion, the water body is divided into two layers at the depth $Z = Z_f$, where the stress τ_x is assumed to vanish. The velocity distribution in each layer is assumed to have a logarithmic profile as given by Prandtl's theory of turbulent flow over rough surface. A different roughness parameter is assumed in each layer. Kivisild verified his theory by comparing it with the data obtained in a hydraulic channel with a depth of 85 cm., as shown in Fig. 3. The theoretical curve seems to fit well with the observed data. However, his theory leads to the infinite velocity at the surface. Also, the assumption of logarithmic profile is valid only when the current is turbulent and the shear is constant with depth. Such conditions are not satisfied in the experiments on which his data were based. The agreement of the theoretical curve with the data does not necessarily imply that his theory is valid because there are two parameters which may be determined arbitrarily. The experimental data seem to agree with the curves computed either by equation (4·7a) or (4·7b) of Ichiye's theory (1952a) with some modification near to the bottom corresponding to the assumption that the current is laminar near the bottom. In fact, the data of Fig. 3 indicates that the height of the bottom layer (Z_f) is only about 0.2 cm. and that the velocity gradient in this layer is steep, suggesting that the bottom layer is a laminar boundary layer.

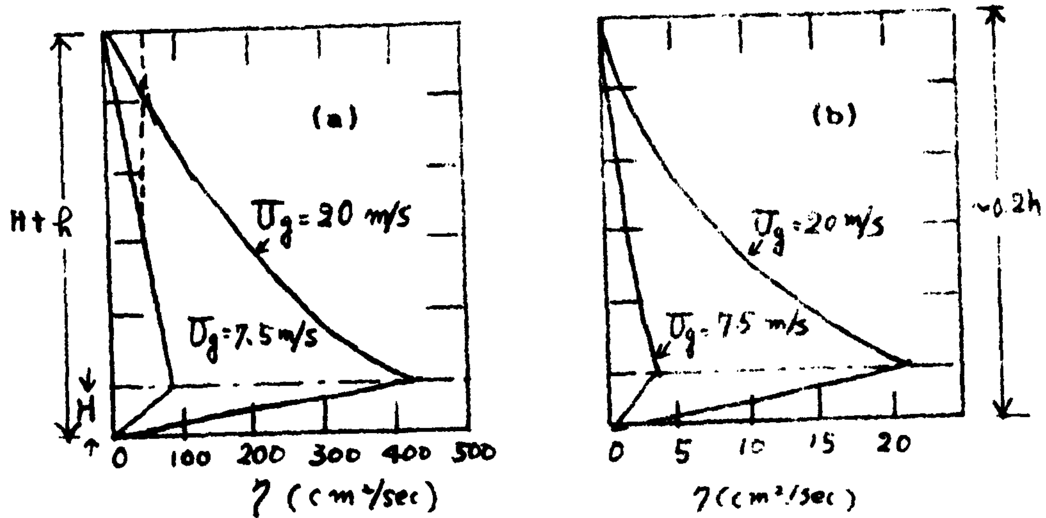

Fig. 1. Vertical distributions of eddy viscosity (η). after Rossby and Montgomery(1935). (z_0=3.2 cm, f=10^{-4}. (a) Adiabatic case. (b) Stable case, with C_1=10 in equation (7.12). See Chapter 7.)

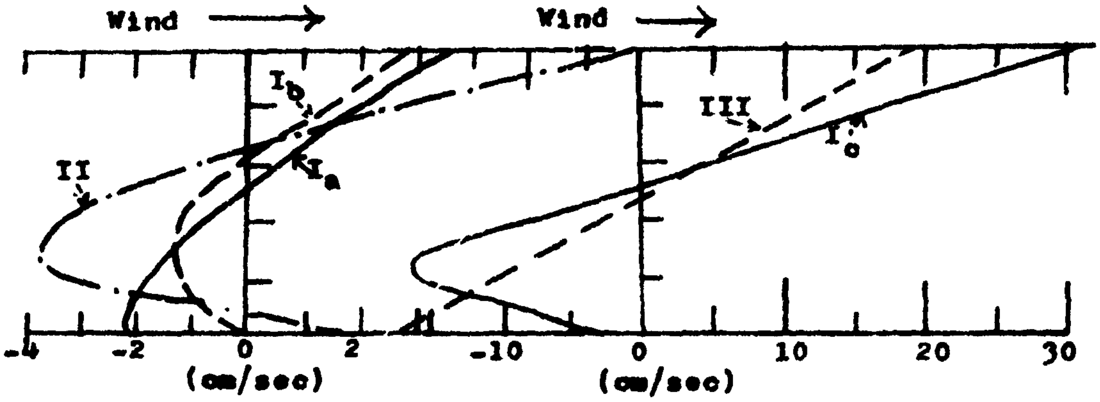

Fig. 2 Vertical distributions of velocity of wind-driven currents in a shallow water, after Ichiye (1952 a). (The curves I, II and III are determined from eqs.(4.7a),(4.7b) and (4.7c), respectively. For I_a, I_b(no current at the bottom) and II, ℓ/h=0.1. For I_c and III, ℓ/h=0.01. Wind stress=1 g/cm sec^2)

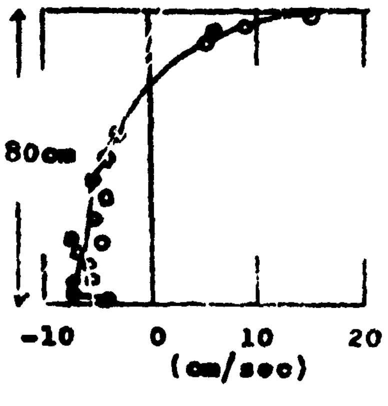

Fig. 3 Velocity profile in a laboratory flume, after Kivisild (1954). (Wind speed at 15 cm above water=10 m/s, z_f=0.2 cm.)

5. Eddy viscosity and storm surges.

Turbulence generated in a shallow water by strong winds is very important to many practical problems like storm surges and beach erosion. However, in the past there was no reliable measurement of turbulence in the sea during a storm. This situation might continue until some instruments which would be sturdy enough to survive the strongest gales and yet sensitive enough to record rapid fluctuations of currents will be constructed.

Storm surges are now studied from two rather indirect approaches. One is the theoretical approach based on either analytical or numerical solution of hydrodynamic equations and the other is the experimental approach using wind tunnels and water tanks.

In the theoretical approach, the equations of motion and continuity are used in a vertically integrated form:

$$\frac{\partial \vec{M}}{\partial t} + f\vec{k}\times\vec{M} + \int_{-h}^{\zeta} \rho\,\vec{u}\cdot\nabla\vec{u}\,dz = -\rho g\nabla(\zeta + h) + \vec{\tau}_s - \vec{\tau}_b \qquad (5\cdot 1)$$

$$\frac{\partial \zeta}{\partial t} + \nabla\cdot\vec{M} = 0 \qquad (5\cdot 2)$$

in which the z-axis is taken positive upward from the origin being at the undisturbed sea level, ζ is the elevation of sea level, h is the depth, $\vec{u}$ is horizontal components of velocity, $\vec{M}$ is mass transport $\left(=\int_{-h}^{\zeta}\rho\vec{u}\,dz\right)$ $\vec{k}$ is an unit vector in z-direction, f is a Coriolis' parameter, ∇ is horizontal

gradient, and $\vec{\tau}_s$ and $\vec{\tau}_b$ are stress vector at the surface and at the bottom, respectively.

In this system of equations, $\vec{M}$, $\vec{u}$, ζ and $\vec{\tau}_b$ are unknown and $\vec{\tau}_s$ is equal to wind stress which can be computed by equation $\vec{\tau}_s = \gamma \rho_a |\vec{W}_a| \vec{W}_a (3 \cdot 21)$. If the integral form of inertia terms and $\vec{\tau}_b$ in equation (5.1) are expressed as functions of $\vec{M}$, ζ and/or $\vec{\tau}_s$, then these equations (5.1) and (5.2) will be solved for $\vec{M}$ and ζ with the suitable boundary conditions which postulate that the mass transport normal to the coast should vanish.

The structure of turbulence in the sea becomes related to dynamics of storm surges through two quantities: bottom stress and integrated inertia terms of equation (5.1). In computation of storm surges from equations (5.1) and (5.2), some assumptions on these two quantities are made. Hansen (1956) assumed that

$$\int_{-h}^{\zeta} \rho \vec{u} \cdot \nabla \vec{u} \, dz = (h+\zeta) \vec{M} \cdot \nabla \vec{M} \qquad (5 \cdot 3)$$

$$\tau_b = const. \vec{M} \, |\vec{M}| \, (h+\zeta)^{-2} \qquad (5 \cdot 4)$$

and computed numerically the tides and currents at the Ems River and the storm surges in the North Sea caused by a storm of January, 1953. Welander (1962) discussed numerical integration of equation (5.1) and (5.2) by taking

$$\tau_b = \mu \vec{M} \qquad (5 \cdot 5)$$

This type of bottom stress was used also by Fischer (1959) in numerical solution of surges in a one-dimensional channel. In his numerical integration

scheme, the bottom stress becomes particularly important, because without this term numerical solution becomes unstable.

In Chapter 4, the mixing length theory is applied to determine velocity distributions and bottom stress of the wind current generated by constant wind stress in a non-rotational uniform basin. Bowden (1953) discussed the bottom stress of the wind-drift in a long channel in which there are strong tidal currents. The bottom current u_b is assumed to be expressed by

$$u_b = u_0 + u_1 \sin \omega t$$

in which u_0 and $u_1 \sin \omega t$ are drift current and tidal current, respectively and $u_1 \gg u_0$. The bottom stress

$$\tau_b = \gamma \rho |u_b| u_b$$

also consists of stationary and non-stationary part. The stationary part τ_b^* becomes

$$\tau_b^* = \rho \gamma' u_0 u_1 \quad , (\gamma' \approx 4\gamma/\pi) \quad (5.6)$$

He determined the value of u_0 and τ_b^* as functions of τ_s , depth h and constant eddy viscosity η for the stationary wind current without surface slopes with a condition that the total transport vanishes. Thus,

$$\tau_b^* = (\tfrac{1}{2}) \tau_s (1 + s_0)^{-1} \qquad (5.7)$$

$$s_0 = 3\eta (\gamma' h u_1)^{-1}$$

The relation similar to this was obtained by Ichiye (1952), although he did not take into account the tidal currents. Weenink (1958) generalized equation (5.7), by adding the term

$$3\rho\bar{u}\eta(1+s_0)^{-1}$$

where $\bar{u}$ is vertically averaged drift current. He applied the generalized formula to the calculation of storm surges in the North Sea.

Fel'zenbaum (1956, 1957) treated the stationary wind currents in a non-rotating shallow water with a variable vertical eddy viscosity. The equations of motion

$$\frac{\partial}{\partial z}\left(\eta\frac{\partial\vec{u}}{\partial z}\right) = -g\nabla\zeta \qquad (5.8)$$

are integrated twice with z, under the boundary condition that bottom current vanishes, giving mass transport as:

$$\vec{M} = -\rho g\nabla\zeta\int_{-h}^{0}\int_{-h}^{z}\frac{z_1}{\eta(z_1)}dz_1\,dz + \vec{\tau}_s\int_{-h}^{0}\int_{-h}^{z}\frac{dz_1\,dz}{\eta(z_1)} \qquad (5.9)$$

in which the surface and bottom are taken at $z=0$ and $z=-h$, respectively, and $\vec{\tau}_s$ is the vectorial wind stress. The integrated equation of continuity; $\nabla\vec{M}=0$, and $\vec{M}$ can be expressed by

$$M_x = -\partial\phi/\partial y, \quad M_y = \partial\phi/\partial x \qquad (5.10)$$

where ϕ is a stream function. Solving $\nabla\zeta$ from equation (5.9) and using the relation $\mathrm{Curl}_z(\nabla\zeta)=0$, we have the equation about ϕ :

$$\nabla\cdot\left(\frac{1}{P}\nabla\phi\right) = \mathrm{Curl}_z\left(\frac{\vec{\tau}_s}{\rho}\frac{Q}{P}\right) \qquad (5.11)$$

in which:

$$P = \int_{-h}^{0} \int_{-h}^{z} \frac{z_1}{\eta(z_1)} \, dz_1 \, dz$$

$$Q = \int_{-h}^{0} \int_{-h}^{z} \frac{dz_1 \, dz}{\eta(z_1)}$$

The boundary condition along the coast is $\phi = 0$. (5·12)

The velocity is now expsessed by

$$\vec{u} = \frac{\vec{M}}{P} \int_{-h}^{z} \frac{z_1}{\eta(z_1)} dz_1 + \frac{\vec{\tau}_s}{\rho} \left[\int_{-h}^{z} \frac{dz_1}{\eta(z_1)} - \frac{Q}{P} \int_{-h}^{z} \frac{z_1 dz_1}{\eta(z_1)} \right] \qquad (5\cdot 13)$$

The wind stress given by (3.21) can be expressed in a vectorial form:

$$\vec{\tau}_s = \gamma \rho_a \vec{W_a} W_a \qquad (5\cdot 14)$$

where $\vec{W_a}$ is the wind vector and $W_a = |\vec{W_a}|$. The eddy viscosity can be expressed by:

$$\eta = C_e W_a h \psi(\bar{z}) \qquad (5\cdot 15)$$

where $\bar{z} = z/h$, following the studies of Sverdrup and Fjeldstad on eddy viscosities of the wind drift in the Arctic Ocean. The constant C_e can be expressed by a more familiar wind factor, ratio of the surface drift to the wind speed. The wind factor C_w is defined in case of the uniform wind flowing over the basin and with such wind, equation (5.8) and boundary condition (5.12) lead to $\phi = 0$. Then,

by taking $\bar{z} = 0$ in equation (5.13) and using relation

$$C_w = |\vec{U}_o| / W_a$$

$$C_e = \frac{\gamma}{C_w \rho}\left[-\frac{\bar{Q}}{\bar{P}} \int_{-1}^{0} \frac{d\bar{z}_1}{\psi(\bar{z}_1)} + \int_{-1}^{0} \frac{\bar{z}_1\, d\bar{z}_1}{\psi(\bar{z}_1)} \right] \tag{5.16}$$

in which

$$\bar{P} = \int_{-1}^{0} \int_{-1}^{\bar{z}} \frac{\bar{z}_1}{\psi(\bar{z}_1)}\, d\bar{z}_1\, d\bar{z} \tag{5.17a}$$

$$\bar{Q} = \int_{-1}^{0} \int_{-1}^{\bar{z}} \frac{d\bar{z}_1\, d\bar{z}}{\psi(\bar{z}_1)} \tag{5.17b}$$

If $\psi(\bar{z}) = (1+\bar{z})^n$, the values of C_e becomes $\gamma/4\rho C_w$, $\gamma/3\rho C_w$ and $2\gamma/5\rho C_w$ for $n = 0$, 1/2 and 3/4 respectively. Then equation (5.11) can be transformed into

$$\nabla \cdot \left(\frac{W_a}{h^2} \nabla \phi \right) = n_w C_w \, \mathrm{curl}_z \left(W_a \vec{W}_a / h \right) \tag{5.18}$$

in which n_w =2, 4 and 8 for $n = 0$, 1/2 and 3/4, respectively. This equation can be solved for ϕ under the boundary condition (5.12). The elevation ζ and velocity are determined from (5.9), and (5.13), respectively.

Gershengorn (1960a) treated the wind-driven currents in an enclosed shallow sea for variable wind stresses, using a constant vertical eddy viscosity η. When the constant wind stress starts suddenly, the steady state of sea levels is established after a time period proportional to h^2/η. If

a parameter $h\,(f/\eta)^{\frac{1}{2}}$ is less than π , the effect of Coriolis' force on the wind current is negligible. Also Gershengorn (1960b) derived a system of ordinary differential equations with time as a variable, by integrating non-linear equations of motion and continuity over an entire body of the enclosed sea. He determined the phase lag between the change of wind stress and response of sea level. It is found that the effect of Coriolis' force becomes negligible for a basin of extremely long or shallow.

With the non-linear bottom stress incorporated, it is found that the period of seiches increases with increasing amplitudes and that the characteristic differences occur in damping in free oscillations , phase shift and modes of resonance for forced oscillations, if the ratio of frictional force to inertia force exceeds a certain critical value.

Ichiye (1950a) compared the mass transports determined from equation (5.1) with the bottom friction of a form (5.5) to that obtained by integration of equations of motion with constant eddy viscosity for stationary wind currents. He derived the expression for coefficient μ in terms of eddy viscosity as:

$$\mu \approx 2\eta\,\left(|\vec{\tau}_s| - \rho g h|\nabla\zeta|\right)\left(|\vec{\tau}_s| - \tfrac{2}{3}\rho g h|\nabla\zeta|\right)^{-1} h^{-2} \qquad (5\cdot 16)$$

in which $\vec{\tau}_s$ is assumed parallel to $\nabla\zeta$ and the depth h satisfies the condition $h \ll (\eta/f)^{\frac{1}{2}}$.

Ichiye (1950b) also derived an equation of the integrated inertia terms in equation (5.1) in terms of mass transport, eddy viscosity, wind stresses and surface slope by substituting

the solution of linearized equations of motion with constant eddy viscosity into $\vec{u}$ of the integral as the first approximation. The result indicates that the approximate formula (5.3) is valid in two extreme cases that the depths are much larger or much smaller than the Ekman's depth of frictional influence, although in the case of deep sea the depth h in (5.3) must be replaced by the depth of frictional influence. He (1950c) applied the same principle to the current generated by a circular wind stress and discussed the vertical motion due to a wind system like hurricanes. Platzman (1962) also used the similar approximation procedure to compute storm surges in Lake Erie by a numerical method. The computed surges show a good agreement with observed data, by adopting the eddy viscosity equal to $40 \ cm^2 \ sec^{-1}$ on the basis of consideration of the decay rate of the lowest mode of the seiches.

Hellstrom (1941) derived velocity profiles of a stationary wind-driven current in a one-dimensional non-rotating channel under various conditions like sloping bottom and three layered fluids using eddy viscosity which is constant with depth. He obtained the relation between the sea level ζ and surface wind stress τ_s :

$$(h+\zeta)\, d(h+\zeta)/dx = \frac{3}{2}\tau_s \qquad (5\cdot20)$$

and tested this relation and velocity profiles, using wind tides data in many lakes and a laboratory experiment. Keulegan (1951) and Francis (1951) also made small scale laboratory investigations on wind-driven currents. Van Dorn (1953) used a

natural pond for studying the relation between wind stress and wind speed by determining the wind set-up.

Sibul (1954) made extensive experiments on the wind tides in a laboratory channel of about 60 feet long, 1 foot wide and 1.15 feet deep. The experiments were conducted with both smooth and rough bottom condition, in order to test the relation (5.18) under various wind speeds. The results indicate that the effect of roughness of the bottom becomes very pronounced in a shallow water, especially for higher wind speed, producing higher set-ups than for a smooth bottom. Tickner (1957) studied the effects of a vegetative roughness on wind-set up using the same channel by varying the depth of the water relatively to the roughness height of 0.1 feet at the bottom. The results indicate that when the water depths were equal to or greater than the roughness height the average set up was higher than for a smooth bottom, but when the water depths were less than the roughness height, the average wind stress becomes progressively smaller, resulting in decrease of set-up. Later Tickner (1960) studied the effects of reefs and bottom slope on wind set-up in the same channel and analysed the data using a modified form of equation (5.18). These experiments, particularly those of Tickner (1957) prove that the turbulence produced by bottom roughness is critically important to the wind set-up in a shallow water. Therefore, ironically they indicate the applicability of the results of laboratory experiments to the situation is doubtful, because the structure of turbulence during a storm in the natural basin is certainly quite different from the one in the laboratory channel.

6. Surface waves and turbulence in the ocean.

It is generally believed that a strong wind agitates the water of the upper layer causing uniform distributions of temperature, salinity and other properties of water. Also, it is considered that not the periodic motion, but the breaking of the waves increases the turbulence of the upper layer of the sea. Although energy transfer from the waves to turbulence through breaking is important, there are very few studies on this problem both in theory and observation. Earlier study of this transfer process was done in quite an abstract form by Ichiye (1952b). A coupling between wave energy and turbulence energy, according to him, is expressed by three relations such as

$$dE(\beta,t)/dt = R_T \pm R_N + R_{t,a} - R_{t,w} \qquad (6\cdot 1)$$

$$d\int_0^\chi F(\chi,t)\,d\chi/dt = \int_0^\chi F(\chi,t)^{\frac{1}{2}}\,d\chi \int_\chi^\infty F(\chi,t)\,\chi^{\frac{1}{2}}\,d\chi \qquad (6\cdot 2)$$

$$R_{t,w} \sim \int_\chi^\infty F(\chi,t)\,\chi^{\frac{1}{2}}\,d\chi \qquad (6\cdot 3)$$

in which $E(\beta,t)$ is a spectrum of wave energy, R_T and R_N are energy transfer due to tangential and normal stresses of wind, respectively. $R_{t,a}$ is a energy transfer due to resonance of waves to turbulent fluctuation of the wind and $R_{t,w}$ is an energy dissipation term caused by the turbulence in the water, and $F(\chi,t)$ is the energy spectrum of turbulence. The argument β in $E(\beta,t)$ is the wave age which is related to χ (wave number of turbulence) by the formula

$$\beta \sim (g/\chi)^{\frac{1}{2}}\,U^{-1} \qquad (6\cdot 4)$$

where U is the mean velocity of the wind at a reference level. Unfortunately, there was no analytical study on the spectrum of wave energy at the time and thus he could not obtain any quantitative result concerning with the coupling.

The problem of energy and momentum transfer from the air to the water was treated in terms of energy spectrum of waves for the first time by Phillips (1957) and Miles (1957). These two authors discussed two basic mechanisms of wave generation separately. The former considered the resonance between the surface wave modes and the pressure fluctuations due to the turbulent wind blowing over the wind. The latter treated the hydrodynamic instability of air-sea interface which was already disturbed. Stewart (1961) discovered from the data on wind waves summarized by Groen and Dorrestein (1958), a fact that the average rate of momentum increase in the waves is almost constant shortly after the onset of wind. He, then, concluded that a large portion of the momentum of the air is transmitted into the water in the form of wave motion, though some portion may go into small waves or ripples which rapidly are dissipated by viscosity.

Phillips (1962) reviewed subsequent development in studies on wave generation suggesting the evolutional mechanism of waves as follows: the wind begins to blow over an initially calm sea, the resonance mechanism is more predominant, until the waves become large enough for the instability to take hold. When the phase velocity c of waves reached the wind speed U, the instability is again very ineffective, while the resonance is operating at its best for the components for which $c \approx U$.

The process of dissipation of wave energy into turbulence through wave energy was also treated by dimensional grounds by Phillips (1958). In the limiting equilibrium state the form of the wave spectrum must be determined by the parameters involved

in the physical process of formation of whitecaps which are generated dynamically by the gravitational instability of the surface: the particle acceleration cannot exceed g. Therefore, the frequency spectrum $\Phi(\omega)$ is dependent only on g and the frequency ω and is given by:

$$\Phi(\omega) = (2\pi)^{-1} \int_{-\infty}^{\infty} \overline{\zeta(t)\zeta(t+\tau)}\, e^{i\omega\tau}\, d\tau \sim g^2\omega^{-5} \qquad (6\cdot5)$$

in which ζ is the surface elevation. Analytical treatment of energy transfer process within energy spectrum of the waves by the nonlinear mechanism was described recently by Hasselman (1962) estimated the energy input from the waves to turbulence, using measurement of turbulence spectrum in the sea, which will be discussed in later section.

More or less empirical studies of eddy viscosity or eddy diffusivity due to wave were done by several authors. Ichiye (1951a) derived the vertical eddy diffusivity η_D in a shallow water as:

$$\eta_D \sim \frac{h_w^2 \sinh(4\pi Z/\lambda)}{T_w \sinh^2(2\pi H/\lambda)} \qquad (6\cdot6)$$

from analogy with Prandtl's mixing length theory, where h_w, λ and T_w are wave height, wave length and period of waves, respectively and H is the depth of the sea. He compared the vertical distributions of suspended materials, determined from diffusion equation with this form of eddy diffusivity with observed data. Later, Ichiye (1953) assumed that the eddy diffusivity in the open sea consists of two parts like:

$$\eta_D = \eta_b + \eta_0 e^{-\alpha Z} \qquad (6\cdot7)$$

in which η_b is the part due to the background turbulence and the exponential t represents the effect of surface waves. He used water temperature data of the upper layer at two Japanese ocean weather stations to determine relationships between the parameters in this eddy diffusivity and the state of the sea s. The empirical formulae obtained by use of least square method from the data for the state of sea from zero to four are

$$\eta_0 / \eta_b = 0.4\, s \qquad (6.8a)$$

$$\alpha = 0.09\, s^{-1} \quad (m^{-1}) \qquad (6.8b)$$

This result is reasonable, because s is proportional to λ (wave length) and h_w (wave height) and also because

$$\eta_0 \sim (h_w^2 / T_w) \sim h_w (\lambda)^{-\frac{1}{2}} \;;\; \alpha \approx 4\pi/\lambda \qquad (6.9)$$

when Prandtl's theory is applied to the turbulence caused by waves. However. numerical values of α determined empirically seem to be much smaller than those given by $4\pi/\lambda$.

In recent years several Russian authors published the results of semi-empirical studies on eddy viscosity due to waves. Dobroklonskii (1947) introduced a formula of the eddy viscosity η_w similar to equation (6·6) assuming that surface waves consist of swells only. He also used Prandtl's mixing length formula like:

$$\eta_w = \ell_w^2 \left|\frac{du}{dz}\right|_w \qquad (6.10)$$

in which ℓ_w and $\left|\frac{du}{dz}\right|_w$ is the mixing length and shear caused by waves. He assumed that the orbital motion of water particles in the surface waves transfer momentum in the vertical direction as turbulent. Then a classical theory of deep water waves yields

$$\ell_w \approx \frac{k_0 \lambda}{6\pi} \;;\; \left|\frac{du}{dz}\right|_w \approx \frac{2\pi^3 h_w^2}{\lambda^2 T_w} e^{-4\pi z/\lambda} \qquad (6\cdot 11a)(6\cdot 11b)$$

where k_0 is Karman's constant and other notations are the same as before. However, analogy between regular orbital motion of waves and irregular motion of turbulence is only superficial, as far as linear motion of the waves is concerned, because it is easily seen that the Reynolds stress $\overline{u'w'}$ vanishes for such orbital motion. Therefore, constants in the expressions for (11a) and (11b) are meaningless, except as determined empirically by comparison with data as done by Ichiye (1951a, 1953a).

Kitaigorodskii (1959a) developed an analogy with mixing length theory further. He assumed that the mixing length consists of two parts: one due to turbulence of surface waves l_w and the other due to turbulence of drift currents l_c. The mean current shear is also divided into the part due to waves $(du/dz)_w$ and the part due to drift currents $(du/dz)_c$. The eddy viscosity is assumed to have a form like:

$$\eta = (l_w + l_c)^2 \left| \left(\frac{du}{dz}\right)_w + \left(\frac{du}{dz}\right)_c \right| \qquad (6 \cdot 12)$$

In this equation he assumed that the quantities related to surface waves are given by (11a) and (11b) and those related to drift currents are by (3·6) and (3·7) after the theory of Rossby and Montgomery (1935) discussed in Section 3. This equation seems to be similar to equation (6·6) derived by Ichiye (1953a), who determined empirically the parameters involved. However, naive application of the mixing length theory as done by Kitaigorodskii seems to be hardly justifiable. For instance, there is no physical ground in the assumption that two parts of mixing length are additive. What is the meaning of a term like $l_c^2 \left(\frac{du}{dz}\right)_w$? The reasonable process to determine η is using the relation between the Reynolds stress $\overline{u'w'}$ and η as given by:

$$-\overline{u'w'} = \eta \frac{\partial u}{\partial z} \qquad (6 \cdot 13)$$

Analytical treatment of this equation in terms of wave characteristics is yet to come. Measurements of Reynolds' stress in the ocean also are very scanty and were obtained only by Bowen (956), whose results will be discussed in a later section.

Kitaigorodskii (1959b) introduced another theory on this problem of generation of turbulence by surface waves. He considered the balance between the energy of waves dissipated by molecular viscosity E_M , and the energy of turbulent flow E_T . The classical theory of viscous flow (Lamb, 1932) yields the equation of the energy dissipation per wave length as:

$$E_M = 2\rho\nu\kappa^3 c^2 a^2 \qquad (6\cdot14)$$

in which κ, a and c are wave number ($\kappa=2\pi/\lambda$; λ is wave length) amplitude and phase velocity of surface waves, respectively and ν is molecular viscosity. Energy dissipation by turbulence per wave length is given from analogy with viscous flow by:

$$E_T = \rho\int_V \overline{\eta\left(\frac{du}{dz}\right)^2} dV \qquad (6\cdot15)$$

where V is the volume of the body of water per wave length and the bar means the average with time. Then he assumed that the eddy viscosity η is expressed by:

$$\eta = l_w^2 \left|\frac{du}{dz}\right|_w \qquad (6\cdot16a)$$

from Prandtl's mixing length hypothesis, in which u is taken to be equal to particle velocity of the waves

$$u = a\omega e^{-\kappa z}\cos(\kappa x - \omega t) \qquad (6\cdot16b)$$

Although the assumption on u is not justifiable, as discussed before, equation (6·15) is valid when proper values of η and

u are used. Kitaigorodskii derived the equation for $[\eta]$, average of η with depth in terms of parameters of surface waves on dimensional grounds. In a steady state the energy of waves dissipated by molecular viscosity is transferred to the energy of turbulence of shear flow which is maintained by wind action. Therefore, we have

$$E_M = E_T$$

From dimensional basis, $[\eta]$ is expressed by the following equation with an assumption that it depends only on λ, wave period T_w and wave height $h_w(=2a)$.

$$[\eta] \sim (h_w)^m \lambda^{2-m} T_w^{-1} \qquad (6 \cdot 17)$$

For further development three elements of waves are replaced by three parameters frequently used in a theory of wave generation: wave steepness $\delta = h_w/\lambda$, wind speed W and wave age $\beta\ (= c/W)$. In terms of the new parameters, equation ( ) becomes:

$$[\eta] \sim \delta^m \beta^3 W^3 g^{-1} = g^{-1} W^3 \varphi(\beta) \qquad (6 \cdot 18)$$

where

$$\varphi(\beta) = [f(\beta)]^m \beta^3 \qquad (6 \cdot 19)$$

because it is confirmed both empirically and theoretically that δ is a function β like

$$\delta = f(\beta) \qquad (6 \cdot 20)$$

Neuman (1952) empirically obtained $f(\beta)$ which is given by:

$$f(\beta) = 0.062 \quad (\text{for } \beta \leq 1/3)$$
$$f(\beta) = 0.1075 e^{-1.667\beta} \quad (\text{for } 1/3 \leq \beta \leq 1) \qquad (6 \cdot 21)$$

Kitagorodsky also quoted a theoretical result of Krylov (1957) who derived: $f(\beta) = 0.06\ (\text{for } \beta \leq 0.375),\ = 0.0225\beta^{-1}\ (\text{for } 0.375 \leq \beta \leq 1)$ $(6 \cdot 22)$

Thus, the problem is reduced to determine m in equation (6·17). Although validity of Prandtl's mixing length theory applied to orbital motion of waves is doubtful, Kitaigorodsky's argument might give some qualitative idea on eddy viscosity due to waves. From empirical evidences two conditions on the behavior of eddy viscosity are postulated by him. The first condition is:

$$\partial [\eta] / \partial \beta \geqq 0 \qquad (6 \cdot 23)$$

in which the equality corresponds to the state of fully developed sea. The second condition is

$$\partial [\eta] / \partial \delta > 0 \qquad (6 \cdot 24)$$

From dimensional ground $[\eta]$ is expressed by

$$[\eta] = h_w \lambda (T_w)^{-1} \psi(\delta) \qquad (6 \cdot 25)$$

Thus, the second condition (24 is replaced by

$$\partial \psi(\delta) / \partial \delta > 0 \qquad (6 \cdot 26)$$

In order to determine m, an assumption on relationships between mixing length and parameters of wave motion is necessary. Three alternative relationships were assumed and tested against the conditions (23) and (24). The first assumption is:

$$\ell \sim h \quad \text{or} \quad \ell \sim h e^{-\kappa z} \qquad (6 \cdot 27)$$

This yields $m = 3$ with use of equations (16a)and (16b). Then, η becomes independent on β, when Krylev's relation (22) is substituted. This might correspond to the state of a fully developed sea. The second assumption is:

$$\ell \sim \lambda \qquad (6 \cdot 28)$$

This, with equations (16a)and (16b) yields $m = 1$. The function $\varphi(\beta)$ now satifies ineqmality in (23). However, the function $\psi(\delta)$ becomes constant, and does not satify the condition (26). The third

assumption

$$\ell \sim h + z \qquad (6\cdot29)$$

This is derived from the theory of [illegible] (1955). Then the function $\psi(\delta)$ is given by:

$$\psi(\delta) = 1 + 3\pi\delta + \frac{9}{2}\pi\delta^2 \qquad (6\cdot30)$$

This function satisfys two conditions (23) and (24) simultaneously, and was considered to be most plausible by Kitaigorodsky.

Also, the vertical distribution of values of eddy viscosity is given by him on dimensional basis as

$$\eta = W^2 g^{-1} \Psi(gzW^{-2}) \qquad (6\cdot31)$$

in which the function $\Psi(X)$ has a form like

$$\Psi(X) = 1 + A_0 X \qquad (6\cdot32)$$

with a positive value of A_0. He considered that the eddy viscosity increases, according to equation (32), from the surface to a certain depth and then decreases rapidly with depth below that level.

Boguslavsky (1957) determined the vertical eddy diffusivity from 16 daily measurements of water temperature and radiation at anchored stations of 200 m deep at a distance of 10 to 12 miles from shore. The formula used for determining eddy viscosity is

$$\eta_\theta(z_1)\,(\partial\theta/\partial z)_{z=z_1} = \int_{z_1}^{H_d} \frac{\partial\theta}{\partial t}\,dz - \frac{1-A_b}{c\rho} I_0 \beta_R \int_{z_1}^{H_d} e^{-\beta_R z}\,dz \qquad (6\cdot33)$$

where θ is the sea temperature, H_d is the depth of the influence of daily temperature variation, A_b is the albedo of the water surface, β_R is the coefficient of radiation absorption and I_0 is the total radiation incident on the sea surface. The relationship between the eddy viscosity and wave conditions is tested by using Dobroklonsky's (1947) theory

$$\eta \approx c_D h_w^2 T^{-1} \qquad (6.34)$$

and Bowden's (1950) theory

$$\eta \approx c_B h_w \lambda T^{-1} \qquad (6.35)$$

where h_w , λ , and T are wave height, wave length and wave period, respectively. The coefficients c_D and c_B are determined from the observed data of h_w, λ and T and the values of η_e computed by equation (6.33). The result indicates that C_B is fairly constant but c_D tends to decrease with increasing wind speed of 1 to 5 m/s.

Shebalin (1957) discussed the eddy viscosity induced by waves in a shallow sea, generalizing the result of Dobronklonsky (1947). He considered the elliptical orbital motion of water particles in linear waves for a finite depth:

$$x = ct - x_1 + h_s \cosh k (H - z_c) \sin \cdot k x_1 \qquad (6.36a)$$

$$z = z_c - h_s \sinh k (H - z_c) \cos k x_1 \qquad (6.36b)$$

where $h_s = \frac{1}{2} h_w (\sinh k H)^{-1}$, x_c and z_c are the coordinates of the center of the orbit, x_1 (= $ct - x_c$) is a coordinate which is moving with the phase velocity c in the direction of positive x , k is the wave number ($= 2\pi/\lambda$),

and H is the depth of the sea. The eddy viscosity is assumed to be given by the mixing length theory $\eta = \rho l^2 \overline{(\partial u/\partial z)_x}$ (6.37a) with mixing length equal to $l = k_o \overline{(\partial u/\partial z)_{x_1}} / \overline{(\partial^2 u/\partial z^2)_{x_1}}$ (6.37b)

where the bar means average along x_1 and over the wave length λ. The shear $\partial u/\partial z$ is given by

$$\frac{\partial u}{\partial z} = - k^2 c\, h_s \frac{\sinh p \cos q - k h_s \sinh p \cosh q}{1 - k^2 h_s^2 (\sinh^2 p + \cos^2 q)} \tag{6.38}$$

where $p = k(H - z_c)$; $q = k x_1$

because $u = dx/dt$ is a function of x_1 and z and z is not explicitly included in the function of u .
Substituting (6·38) and similar expression for $\partial^2 u/\partial z^2$ into the formula (6·37a,b) and neglecting small terms, we have

$$\eta \approx \frac{k_0^2 \pi \rho h_\omega^2}{36 T} \frac{\sinh^3 2k(H - z_c)}{\sinh^2 kH \cosh^2 2k(H - z_c)} \tag{6.39}$$

When the sea is very deep, this eqution becomes

$$\eta \approx \frac{k_0^2 \pi \rho h_\omega^2}{18 T} e^{-4\pi z_c/\lambda} \tag{6.40}$$

which becomes equal to the equation derived by Dobroklonsky(1947).

Groen (1954) considered the motion due to deep sea waves consisting of three parts: the pure wave motion, the external turbulent motion $(\vec{u}^*)$ which exists independently of the wave, and the secondary motion induced by coupling of the above two motions. He derived the energy equation over one wave length which includes the horizontal eddy viscosity in a form of $\overline{u^* k^{-1} \sinh k\xi}$ as well as the vertical viscosity, where k is a wave number and ξ is a orbital displacement. He then considered that the vertical vis-

cosity is much smaller than the horizontal viscosity which is expressed by $\lambda^{3/4}\ \varphi(2\pi z/\lambda)$ from Richardson's 4/3-power law (Chapter 10), where φ is a function to be determined from empirical data.

The turbulence due to wave action is generated by non-linear processes, for the most part breaking of waves and to the lesser degree the energy transfer between different wave components. The works reviewed in this chapter are heuristic in their argument and a more rigorous treatment of the problem is yet to come. A speculative treatment by Stewart (1962) of the relationship between the turbulence and wave breaking will be discussed in chapter 9. Hasselman (1962) treated the energy transfer resulting from weak non-linear couplings between the spectral components of waves by means of a perturbation method. He obtained the energy flux transferred from three "active" wave components to a "passive" fourth component as a result of the fifth order analysis. He assumed the Gaussian distribution of the wind generated random sea and dismissed the statistical process which is important in the energy transfer in the turbulence. The analysis involved is still too complicated to be related to the actual phenomena.

7. Effect of stability

The stratification of the lower atmosphere is as often unstable as stable. This thermal instability causes not only regular, cell-like convection, but also increases the turbulence. Therefore, turbulence and transfer of heat and moisture in the lower atmo[illegible] are to a great degree dependent on the stability. Many meteorologists worked for a long time on this problem and their results are summarized by Priestly (1959) in his well wr[illegible] book. In this chapter, essential points of the progres[illegible] ved in meteorology is briefly discussed.

Ri[illegible] introduced the criterion on the growth of turbulenc[illegible] ble fluid. This criterion says that the turbulence grows or decays according as the rate of energy consumed by fluid particles moving against stability (buoyancy or Archimedes forces) is less or greater than the rate of energy supplied to turbulence by the mean shearing motion. Therefore, the ratio of rates of consumed and supplies energy represent in a sense the intensity of turbulence. This ratio is called the flux Richardson number and expressed for the atmosphere by

$$R_f = - \frac{g \Theta}{C_p \theta \tau \ (\partial \bar{u} / \partial z)} \qquad (7 \cdot 1)$$

where Θ is the heat flux and τ is the shearing stress of the wind. These quantities are given by:

$$\tau = \rho_a \eta_M \partial \bar{u} / \partial z \qquad (7 \cdot 2)$$

$$\Theta = - \rho_a c_p \eta_\theta \ (\partial \bar{\theta} / \partial z + \Gamma) \qquad (7 \cdot 3)$$

in which η_M and η_θ are eddy viscosity and eddy diffusivity, respectively, and θ is the temperature and Γ is the adiabatic lapse rate. Negative values of R_f mean an unstable condition. This R_f reduces to the more familiar <u>gradient</u> Richardson number

$$R_i = \frac{g}{\theta} \frac{(\partial \theta/\partial z)+\Gamma}{(\partial u/\partial z)^2} \qquad (7\cdot 4)$$

multiplies by η_θ / η_M.

Monin and Obukhov (1953) developed the similarity theory of the turbulence in the lower atmosphere and introduced the scale of length L characterizing the stability condition,

$$L = - \frac{u_*^3 \, c_p \rho_a \theta}{k_o g \Theta} \qquad (7\cdot 5)$$

in which u_* is the friction velocity $(=\sqrt{\tau/\rho_a})$. This length scale becomes infinite under neutral stratification and is positive or negative under stable or unstable condition, respectively. The mean wind profile can be expressed by a function of the dimensionless height z/L,

$$\bar{u}(z) = \frac{u_*}{k_o} \left[F(z/L) - F(z_o/L) \right] \qquad (7\cdot 6)$$

in which z_o is the roughness parameter and $F(z/L)$ is an universal function. There is a close relationship among R_i, R_f and z/L.

The similarity theory of Karman (which was also used in the mixing length theory) gives the velocity profile and eddy viscosity for a neutral stability,

$$\frac{\partial u}{\partial z} = \frac{u_*}{k_o z}, \quad \eta_M = k_o^2 z^2 \left|\frac{\partial u}{\partial z}\right| = u_* k_o z \qquad (7\cdot 7a, b)$$

When this theory is generalized to include the effect of stability, the modified velocity profile is given by:

$$\frac{\partial u}{\partial z} = \frac{u_*}{k_0 z} f(\xi) \qquad (7.8)$$

in which ξ is any one of parameters R_i, R_f and z/L and is another universal function. When ξ is taken equal to $z/L (= \xi_L)$, comparison of equations (7.6) and (7.8) yields

$$f(\xi_L) = F'(\xi_L)\,\xi_L \qquad (7.9)$$

The modified eddy viscosity is given by:

$$\eta_M = \eta_0\, f(\xi) = u_* z\, \eta_M^* \qquad (7.10)$$

in which η_0 is the eddy viscosity for a neutral condition and η_M^* is a dimensionless form of η_M or can be regarded as a generalized Karman's constant $(= k_0 f(\xi))$. The relationship between R_i, R_f and z/L is given by:

$$\frac{\eta_\theta}{\eta_M} R_i = R_f = \frac{\eta_M^*}{k_0}\left(\frac{z}{L}\right) \qquad (7.11)$$

The form of the function $f(\xi)$ for taking R_i as ξ was derived from energy consideration by Rossby and Montgomery (1935) and is equal to:

$$f(R_i) = (1 + C_1 R_i)^{-\frac{1}{2}} \qquad (7.12)$$

in which C_1 is a positive constant. This form will fail in the unstable regime. Holzman (1943) has introduced another form,

$$f(R_i) = (1 - C_2 R_i)^{\frac{1}{2}} \qquad (7.13)$$

This form will fail at large stabilities but is preferred by meteorologists to equation (7.12).

The ocean surface supplies heat to the atmosphere both in latent and sensible form in an averaged condition over the seasons and over the whole ocean. Therefore, the stratification of the upper layer of the ocean is, as a whole, stable. Also, mechanical stirring of wave action is predominant over convective mixing in most parts of the world oceans even in cooling seasons. However, classical hydrographic measurements by Jacobsen (1913) of current and density distributions in Norwegian fjords stimulated the interest of Taylor (1931). According to the original mixing length theory of Prandtl, eddy viscosity and eddy diffusivity is equal and thus R_f is also equal to R_i. Thus, from Richardson's criterion turbulence must be suppressed for R_i larger than one. However, Taylor found that the shear stress and transport of salinity measured by Jacobsen were much larger than could be accounted for by molecular viscosity and diffusion for R_i as high as 30. He computed the values of η_M and η_S from the shear and salinity transport by use of relations (7·2) and (7·3) and concluded that η_S is larger than η_M for turbulent flow. The range of computed values of η_S/η_M, R_i and R_f for turbulent conditions are 0.02 to 0.20, 3 to 30 and 0.2 to 1.3, respectively.

Townsend (1957) recently discussed the interaction between the temperature and velocity fields using the equations for the turbulent intensity and for the mean square temperature. He concluded that the critical values of R_f and R_i are 0.5 and 0.08, respectively and that the turbulent motion would collapse if these values are exceeded. This conclusion was confirmed by

the experiments of heated surface in a wind tunnel and of a jet injected along the interface between saline solutions of different densities. He explained that the large Richardson number obtained by Taylor is due to the irregular waves of the interface which cause transport of momentum and salinity much larger than the molecular rates and also produce momentum transfer more intense than salinity transport.

Munk and Anderson (1948) discussed the mechanism of development of the upper homogeneous layer and thermocline by wind action. They considered the steady state, in which the momentum transfer downward from the wind stress is prevented by the decrease of vertical viscosity at the thermocline. First, they used the eddy viscosity of the form (7.12) introduced by Rossby and Montgomery (1935). Also, they adopted the value of C_1 equal to 10 according to Sverdrup (1936), who estimated it from the wind speed measurements over the snow field. In order to determine the eddy diffusivity which is assumed to be the same for temperature and salinity, they utilized two relations obtained by Jacobsen (1913) and Taylor (1931) which are discussed above. These are

$$\eta_S / \eta_M \leq 1 \tag{7·14 a}$$

and

$$R_f = R_i(\eta_\theta / \eta_M) \leq 1 \tag{7·14 b}$$

In the latter relation they postulated that R_f should reach unity assymptotically as R_i increases infinitely. These condition yield:

$$\eta_s = \eta_0 \left(1 + \frac{1}{3} C_1 R_i\right)^{-3/2} \qquad (7\cdot15)$$

In order to obtain velocity and temperature distributions, they started from the equations of motion of Ekman type.

$$-f v = \frac{d}{dz}\left(\eta_M \frac{du}{dz}\right) = \frac{d}{dz}(\tau_x) \qquad (7\cdot16a)$$

$$f u = \frac{d}{dz}\left(\eta_M \frac{dv}{dz}\right) = \frac{d}{dz}(\tau_y) \qquad (7\cdot16b)$$

for the velocity distribution. The equation of heat transfer was taken as:

$$\Theta = -c_p \rho \eta_\theta \frac{\partial \theta}{\partial z} \qquad (7\cdot17)$$

and Θ was assumed to be constant with depth. The Richardson number is now defined as

$$R_i = g\alpha \frac{\partial \theta}{\partial z} \Big/ (\partial u/\partial z)^2 \qquad (7\cdot18)$$

in which α is a generalized thermal expansion coefficient given by

$$\alpha = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial \theta} + \frac{dS}{d\theta}\frac{\partial \rho}{\partial S}\right) \qquad (7\cdot19)$$

under the assumption that temperature and salinity (S) has a definite correlation.

Since equation (7.16) as the first approximation is the second order differential equation of z, two boundary conditions are necessary. They took both conditions for the surface,

contrary to Ekman. These conditions are:

$$\tau_x = 0 \quad ; \quad \tau_y = \tau_s = \text{Const.} \qquad (\text{at } z=0) \qquad (7.20a)$$

$$u = v = \tau_s/(\eta_0 f)^{\frac{1}{2}} \qquad (\text{at } z=0) \qquad (7.20b)$$

in which τ_s is the wind stress. The second condition comes from the assumption that the surface currents are the same as derived by Ekman under the assumptions of constant eddy viscosity η_0 and of infinite depth.

The equations (7.16) and (7.17) are a system of non-linear differential equations about u, v and θ for the variable z. These were solved by numerical integration. It was assumed that η_0 is given by the relation of Ekman (Sverdrup et al, "The Oceans" 1942:496) $\eta_0 = 4.3\,W^2$ (W in cm/sec, $W > 6$ m/sec) and that the wind stress is almost equal to $3.6 \times 10^{-6}\,W^2$. The result computed is shown in fig. 4. The curves indicate that the stress, eddy viscosity and diffusivity reach the minimum near the depth of maximum temperature gradient. They admitted that the solution becomes inadequate the greater the depth owing to the assumption of constancy of Θ with depth.

The parameters which determine the distributions of temperature and currents are wind speed W, heat flux Θ, latitude and generalized thermal expansion coefficient α. The calculations showed that the wind speed is most effective in changing the distributions. Doubling wind speed causes thermocline lower from 22 to 61 meters Doubling the latitude raises the

thermocline by 40% doubling the heat flux by 2% and doubling α by only 15%.

Mamaev (1958) criticized the assumptions of Munk and Anderson on the form of η_M and η_S. First, the eddy viscosity assumed by them will fail in an unstable condition, for which the values of R_i is less than -0.1. Second, according to Richardson's criterion, the flux Richardson number R_f must vanish for infinite value of R_i, because the generation of potential energy is inhibited by increasing stability. However, Munk and Anderson's theory yields $R_f = 0.52$ for $R_i = \infty$. In an unstable condition the convective mixing due to instability tends to equalize only the density along the vertical direction but the mean current is less affected by instability. Therefore, finally the condition $\eta_M \geqslant \eta_S$ must be satisfied for $R_i < 0$.

He suggested the form for eddy viscosity and diffusivity as:

$$\eta_M = \eta_0 e^{-m R_i} \qquad (7\cdot 21a)$$

$$\eta_\theta = \eta_0 e^{-n R_i} \qquad (7\cdot 21b)$$

From the second and the third condition, there must be a relation $n - m > 0$. This new form is valid for all values of R_i. The eddy viscosity and diffusivity determined from Jacobsen's data (1913) are expressed by taking $m = 0.4$, $n = 0.8$. When $n - m \leq e^{-1}$, R_f becomes larger than one in a certain range of R_i, in which the turbulence is damped. He considered that the occurrence of the regime of damping turbulence is due to a fact that the disturbances in this range are of oscillatory type having a certain range of wave lengths.

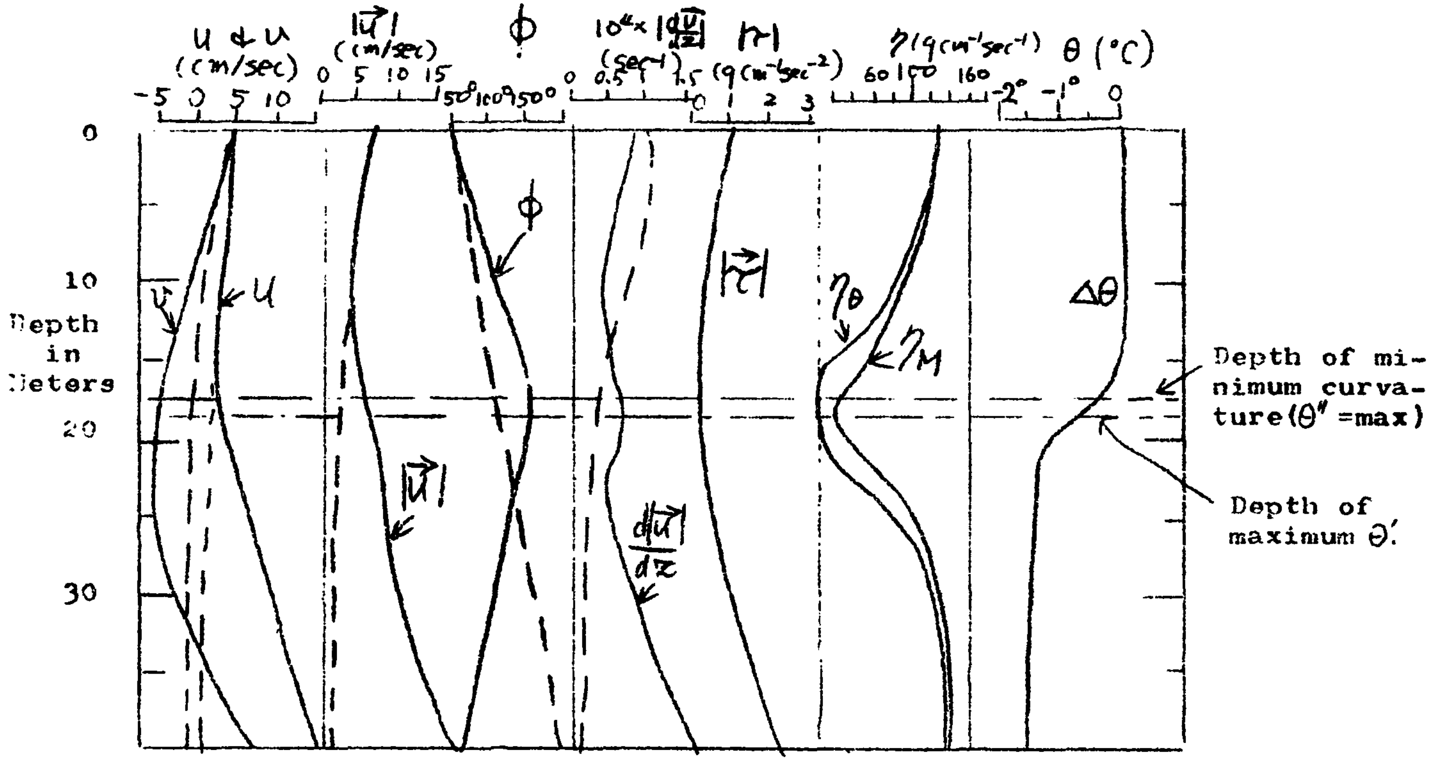

Fig.4. An example of vertical distributions of velocity of wind -driven current u and v, its magnitude, $|\vec{u}|$, deviation from the wind direction, ϕ current shear, $\frac{d|\vec{u}|}{dz}$, shearing stress, $\vec{\tau}$, eddy viscosity and diffusivity η_M, η_θ, and temperature, θ .(Munk and Anderson, 1947) (β $\alpha = 2\times10^{-4}$(o C), $\tau_s = 1.02$, (g $cm^{-1} sec^{-2}$), $\eta_0 = 155$ g $cm^{-1} sec_{-}^{-1}$, Ⓗ $= 2\times10^{-3}$ cal $sec^{-1} cm^{-1}$.)

Later Mamaev (1959) proposed a method of determining the depth of no motion in the dynamic calculation of ocean currents as the depth where the total energy transfer from the mean shearing flow is equal to the total work done by Archimedes forces. Thus, this depth H_n is determined from the relation:

$$\int_0^{H_n} \eta_M \left(\frac{\partial u}{\partial z}\right)^2 dz = \int_0^{H_n} \eta_S E_S \, dz \qquad (7.22)$$

where the surface is taken at $z=0$ and E_S is the stability. Assuming that the current and density distributions are linear with depth, he derived:

$$H_n = \left(\frac{\eta_M}{\eta_S}\right)^{\frac{1}{2}} u(0) \left(g \overline{E}_S\right)^{-\frac{1}{2}} \qquad (7.23)$$

in which $\overline{E}_S$ is the mean stability given by:

$$\overline{E}_s = \left[\rho(H_n) - \rho(0)\right] / \overline{\rho} H_n$$

He compared the depth of no motion H_n determined by Defant (1941) with $(g\overline{E}_s)^{\frac{1}{2}}$ at 224 "Meteor" stations and obtained the empirical formula:

$$H_n (g\overline{E}_s)^{\frac{1}{2}} = 4.5 \ m/sec \qquad (7.24)$$

He suggested this relation to be used for estimating the depth of no motion.

In order to determine the depth of upper homogeneous layer h of the ocean, Kitaigorodsky (1960) introduced the idea similar to that of Monin and Obukhov (1953) on the length L in the lower atmosphere. When Coriolis' force is neglected, parameters to be considered are the wind stress τ, heat transfer at the

surface Θ_0, thermal expansion coefficient α and gravity g. From dimensional grounds, he proposed the relation:

$$k = C_k \tau^{3/2} (g\alpha\Theta_0)^{-1} \qquad (7.25)$$

in which C_k is a numerical constant. He determined τ from Neuman's formula, Θ_0 from Budyko's chart, and k and α from the NORPAC data and obtained $C_k = 2.00$. When Coriolis' force is considered, dimensional analysis leads to the condition that C_k is a function of a non-dimensional parameter

$$N_k = \tau f (g\alpha\Theta_0)^{-1} \qquad (7.26)$$

From the same data used in determining C_k, he obtained the relation

$$C_k \sim 1/N_k \qquad (7.27)$$

When the wind stress is assumed to be proportional to the square of the wind speed, this relation with the definition of k leads to a formula:

$$k \sim W (f)^{-1} \qquad (7.28)$$

which was derived by Rossby and Montgomery (1935) (See Eq. (3.20).)

The theory of thermocline as a global phenomena was recently treated by several authors not from a point of view of turbulence problem, but of thermohaline circulation of the ocean. Ichiye (1958a) introduced a model of density distributions based on such an idea. When the vorticity equation of planetary motion is differenciated with z and the equation of continuity and the geostrophic relation are substituted, we have:

$$\partial^2(\rho w)/\partial z^2 + (\beta g f^{-2})\partial\rho/\partial x = 0 \qquad (7.29)$$

when frictional terms are neglected. In this equation the x-axis is taken eastwards and β is the latitudinal change of f. The equation of transport of mass becomes

$$w \frac{\partial \rho}{\partial z} = \eta_\rho \frac{\partial^2 \rho}{\partial z^2} \qquad (7.30)$$

since the geostrophic assumption for horizontal currents makes the advection terms almost cancel each other. The result indicates that the density increasing eastwards causes upward motion and creates sharp thermocline near the surface, while the density decreasing eastwards causes downward motion and makes thermocline deeper. Robinson and Stommel (1959) discussed the similar problem, by using temperature in place of density. Instead of (7.29) and (7.30) they used the differentiated vorticity equation

$$\partial^2 w/\partial z^2 - (\alpha \beta g f^{-2})\, \partial \theta/\partial x = 0 \qquad (7.31)$$

and the mass transport equation

$$\eta_\theta\, \partial^2 \theta/\partial z^2 - w\, \partial \theta/\partial z - f(\partial w/\partial z)(\partial \theta/\partial y) \qquad (7.32)$$

in which they retained the advection term of the latitudinal direction. Both of these studies indicate that the change in the depth of the thermocline is dependent on the vertical motion of a grobal scale and the effects of vertical eddy diffusivity on the thermocline is, if any, of a minor role.

There are two studies which treated the effects of both waves and stability on turbulence. Kent and Pritchard (1959) considered the mixing length of the form:

$$l = l_0 (1 \pm c_j R_i)^n \qquad (7.33)$$

in which ℓ_0 is the value corresponding to the neutral stability. The constant n and double signs were determined by different authors: $n = -\frac{1}{2}$ and plus sign by Rossby and Montgomery (1935), $n = \frac{1}{2}$ and minus sign by Holzman (1943) and $n = -1$ and plus sign as another alternative. Then, they applied the generalized mixing length defined by Montgomery (1943) for a cylinder of any cross section to an estuary of the finite depth H. Montgomery's definition is:

$$\ell_0 = k_0 \left[\frac{2}{\int_0^{2\pi} \frac{d\phi}{s(\phi)}} + z_0 \right] \qquad (7.34)$$

where $s(\phi)$ is the distance in the direction ϕ from a point considered to the wall. This expression of ℓ becomes for the estuary $\ell_0 \sim k_0 z(1 - z/H)$. They further assumed that the effect of surface waves decrease with depth as $\exp(-2\pi z/\lambda)$, where λ is the wave length. Combining these two effects, they got

$$\ell_0 = k_0 z (1 - z/H)(1 + a e^{-2\pi z/\lambda}) \qquad (7.35)$$

where a is the constant to be determined.

They determined the mixing length ℓ from the data of fluctuations of a horizontal velocity along the estuary and salinity, assuming that

$$[\overline{(S')^2}]^{\frac{1}{2}} = \ell_S \frac{\partial \bar{S}}{\partial z}, \qquad [\overline{(u')^2}]^{\frac{1}{2}} = \ell_M \frac{\partial \bar{u}}{\partial z} \qquad (7.36\ a,b)$$

where the primed and barred quantities represent the fluctuations and mean values, respectively. They used the geometrical mean of ℓ_S and ℓ_M as the observed mixing length and compared with the form derived from theoretical consideration. However, their results are hardly reliable with so many unproved assumptions involved.

Kitaigorodsky (1961) discussed first the temperature distributions due to the turbulence caused by wave action only and then the effect of stratification on the wave-generated turbulence. According to his earlier study reviewed in the previous section the orbital motion and the averaged (or integral) eddy viscosity, are from dimensional basis given by:

$$\overline{u}_{orb} \sim \delta (\bar{\lambda})^{\frac{1}{2}} g^{1/2} e^{-z/\bar{\lambda}} \qquad (7.37)$$

and

$$\overline{\eta} \sim g^{\frac{1}{2}} (\bar{\lambda})^{3/2} \delta^{\alpha-2} \qquad (7.38)$$

(δ =wave steepness= h_w/λ ; h_w ,wave height, λ wave length) respectively, in which $\bar{\lambda} = \lambda/2\pi$ and the period is replaced by wave length. The constant α depends on the hypothesis for mixing length or "turbulency scale" by his terminology and it is equal to 3 for the most plausible hypothesis

$$l = z + \frac{1}{2} h_w \qquad (7.39)$$

When the stationary state is reached under the stirring action of waves, the physical parameters which determine the vertical profile of temperature, are heat flow $\textcircled{H}$, specific heat capacity C_p , density ρ , g and wave length λ , if the stratification is almost adiabatic. The dimensional and similarity consideration yield:

$$d\theta/dz \sim \theta^* \lambda^{-1} \Phi(\epsilon) \qquad (7.40)$$

in which θ^* and ϵ are the characteristic temperature:

$$\theta^* \sim - \textcircled{H} (c_p \rho)^{-1} (g\lambda)^{-\frac{1}{2}} \qquad (7.41)$$

and a dimension-less vertical coordinate $\epsilon = z/\bar{\lambda}$, respectively and $\Phi(\epsilon)$ is an universal function. Assuming that

the eddy diffusivity η_D is given by:

$$\eta_D \sim \eta \simeq \ell^2 \left(\frac{d\,\overline{u}_{orb}}{dz}\right) \qquad (7.42)$$

he derived:

$$\eta_D = K_D \delta g^{1/2} (\bar{\lambda})^{3/2} e^{-z/\bar{\lambda}} \left(\frac{2}{\bar{\lambda}} + \pi\delta\right)^2 \qquad (7.43)$$

by using the assumption that $\ell \sim z + \frac{1}{2} h_w$, where K_D is the unknown numerical constant. For developed turbulence it is shown that the ratio η / η_D is almost equal to the molecular Prandtl number, which becomes 10 for the sea water. Comparing the heat transport equation:

$$\eta_\theta \frac{d\theta}{dz} = - \frac{\textcircled{H}}{\rho c_p} = const. \qquad (7.44)$$

with equation (7.40), he determined the functional form of $\Phi(\epsilon)$. The integration of equation (7.44) with z yields the temperature distribution:

$$\theta(\epsilon) - \theta(0) = \theta_* \left[\varphi(\epsilon) - \varphi(0) \right] \qquad (7.45)$$

in which $\varphi(\epsilon)$ is defined by

$$\varphi(\epsilon) = \int_{\epsilon_0}^{\epsilon} e^{y} y^{-2} dy$$

and $\epsilon_0 = \pi\delta$. The value of θ_* is now expressed by:

$$\theta_* = (K_D \delta)^{-1} e^{-\pi\delta} = - \textcircled{H} (\rho c_p)^{-1} (g\bar{\lambda})^{-\frac{1}{2}} \qquad (7.46)$$

When stability is taken into account, the universal function $\Phi(\epsilon)$ must be replaced by $\Phi_1(\epsilon, M_s)$, in which M_s is the new dimension-less parameter.

$$M_s \sim |\textcircled{H}| \alpha (\rho c_p)^{-1} (g\lambda)^{-\frac{1}{2}} \qquad (7.47)$$

and α is the temperature expansion coefficient $(= \rho^{-1}\partial\rho/\partial\theta)$. It is evident that $\Phi_1(\epsilon, M_s) \rightarrow \Phi(\epsilon)$ at $M_s \rightarrow 0$ (i.e., when the buoyancy forces can be neglected). The "turbulence scale"

l is a function of Richardson number, which is expressed, for the wave generated turbulence as:

$$R_i = g\alpha \frac{d\theta}{dz}\left(\frac{d\,\bar{U}_{orb}}{dz}\right)^{-2} \qquad (7.48)$$

If the formula of l is assumed to be:

$$l \sim (z + \tfrac{1}{2}h_w)\, f(Ri) \qquad (7.49)$$

where $f(R_i) = (1 + C_1 R_i)^{\frac{1}{2}}$ according to Rossby and Montgomery (1935), the calculation similar to the neutral stability yields the temperature distribution, universal function $\Phi_1(\epsilon, M_s)$ and M_s:

$$\theta(\epsilon) - \theta(0) = \Theta_*\left[\varphi_1(\epsilon, M_s) - \varphi_1(0, M_s)\right] \qquad (7.50)$$

$$\varphi_1(\epsilon, M_s) = \int_{\epsilon_0}^{\epsilon} e^{y}\left(y^2 - M_s e^{3y}\right)^{-1} dy \qquad (7.51)$$

$$M_s = B_s\,|\Theta|\,\alpha\,(\rho C_p)^{-1}(g\bar{\lambda})^{-\frac{1}{2}}\,\delta^{-3} e^{-3\pi\delta} \qquad (7.52)$$

where B_s (>0) is the numerical constant.

He determined the vertical distributions of temperature in the upper layer of the sea, using equations (7.48) and (7.50). There is an empirical relation between steepness δ and wave age c/W (c is wave velocity and W is the wind speed). For the fully developed seas, $c/W = 0.7$ and $\delta = 0.032$. The wave length for this state is a function of wind velocity and is given by:

$$\lambda = 2\pi c^2 g^{-1} \sim 3.1\, W^2 g^{-1} \qquad (7.53)$$

Therefore the temperature distributions in the fully developed seas are determined from equation (74) and (75) when $\overline{W}$ and $\textcircled{H}$ are known. The results of such calculations are shown in fig.5 by taking $\textcircled{H} = 6.9 \times 10^{-3} g\,sec^{-3}$, $K_D = 2.0 \times 10^{-2}$ $B_S = 6.0 \times 10^{-3}$ and $\alpha = 6.0 \times 10^{-3}$.

The curves indicate that the wave-generated turbulence creates the upper homogeneous layer with thickness of the order of one wave length, that this thickness increases with the wind speed, that there occurs a sharp thermocline below the homogeneous layer and that the effect of stability causes a more abrupt gradient of temperature in the thermocline.

He suggested that, instead of wind velocity $\overline{W}$, a frictional velocity of wind W_* $(= \sqrt{\tau / \rho_a})$ should be used, since the latter is a more objective characteristic of wind. Then parameters, θ^*, ϵ and M_S can be expressed with W_* in place of λ from the relation $\lambda \sim W_*^2 g^{-1}$.

The wave age c/W_* depends either on the dimensionless fetch gF/W_*^2 for a limited fetch F or on the dimensionless duration gT_d/W_* for a limited duration T_d. For the fully developed seas in an unlimited fetch, c/W_* is a certain universal constant and thus all these three parameters are determined if two parameters W_* and $\textcircled{H}$ are known.

He further discussed the validity of several assumptions involved in his theory. He argued that the assumption of uniform steepness is permissible if we take this steepness as

an averaged value from the observed spectrum of waves. Also, the assumption of taking only $\overline{U_{orb}}$ as the energy supplying mean shear flow is valid because the mean transport flow (drift current) of Stokes' type and its vertical gradient in the distance of order of ℓ are much smaller than the orbital velocity and its gradient. He also argued that the mixing in fully developed seas is mostly caused by small scale turbulence ($\ell \ll \lambda$) considered here, because in such state a large part of wave energy is transported by waves propagating with a speed close to that of wind and is not used for mixing.

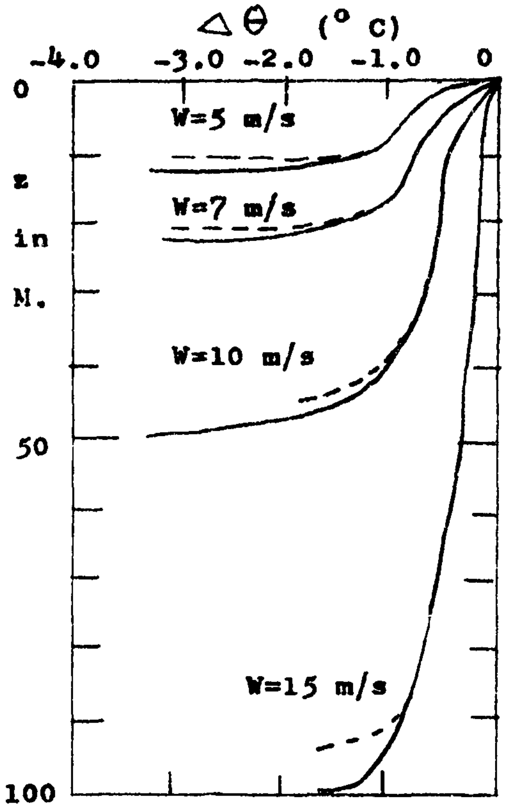

Fig. 5. Vertical profiles of water temperatures for different wind speeds. (Kitaigorodsky, 1961.) (c/W= 0.7, Θ =6.9x10^{-3} g cal/sec^3, λ =8, 14, 32 and 80 cm for W=5, 7, 10 and 15 m/s, respectively.)

Ellison and Turner (1959) introduced the concept of entrainment in a flow system in which a lighter fluid is emitted by a source under a sloping roof (or a heavier fluid on a sloping floor). In the plume or jet, x- and Z- axis are taken along and perpendicular to it, respectively and $u(x, z)$ is its mean velocity relative to that in the ambient fluid. Its velocity, length and density scales, $U(x)$, $h(x)$ and $\Delta(x)/g$, in the cross section and the flux of density difference (independent on x) A are defined by

$$U h = \int u(x,z)\,dz \,, \quad U^2 h = \int u^2(x,z)\,dz \qquad (7.54\ a,b)$$

$$A = \int \left(\frac{\rho_b-\rho}{\rho_b}\right) g\,(u+u_b)\,dz = (U+U_b)\,h\,\Delta \qquad (7.54\ c)$$

where the suffix b refers to the quantities of the ambient fluid and z is measured from the boundary (roof or bed) for the flume but it is measured from the plane of symmetry [illegible] the jet.
The entrainment constant E is defined by:

$$d\,[h\,(U+U_b)]\,/dx = E\,|U| \qquad (7.55)$$

In the case of a lighter fluid flowing up a roof over a stationary ambient fluid $(U_b=0)$, in which the pressure is hydrostatic, the equations for continuity and momentum transport become:

$$A_1 = U h \Delta = \text{const.} \qquad (7.56)$$

$$\frac{d(U^2 h)}{dx} = -\,C\,U^2 - \frac{1}{2}\,\frac{d(S_1 \Delta h^2 \cos a)}{dx} + S_2 \Delta h \qquad (7.57)$$

where

$$S_1 h^2 \Delta = \int_0^{\infty} 2g\left(\frac{\rho_b-\rho}{\rho_b}\right) z\,dz \,, \quad S_2 h \Delta = \int_0^{\infty} g\left(\frac{\rho_b-\rho}{\rho_b}\right) dz \qquad (7.58\ a,b)$$

and a is the slope of the plume. Equation (7.57) represents the momentum balance of the fluid contained in the volume bounded by

x, $x+dx$; $y=0$, $y=\infty$. The first term on the right-hand side is the frictional drag of the roof, the second expresses the difference of the integrated hydrostatic pressure on the planes x and $x+dx$ and the last term represents the gravity force. The drag coefficient C is not constant but varies with the shape of the profiles and the stability near the wall.

Introducing the overall Richardson number

$$R_i = \cos a \cdot h\Delta / U^2 = A \cos a / U^2(U+U_b) \qquad (7.59)$$

and using the coefficient of entrainment of equation (7.55) with $U_b = 0$, equation (7.57) leads to separate equations for dh/dx and dR_i/dx :

$$\frac{dh}{dx} = \frac{(2-\frac{1}{2}S_1 R_i)E - S_2 R_i \tan a + C}{1 - S_1 R_i} \qquad (7.60\,a)$$

$$\frac{h}{3R_i}\frac{dR_i}{dx} = \frac{(1+\frac{1}{2}S_1 R_i)E - S_2 R_i \tan a + C}{1 - S_1 R_i} , \qquad (7.60\,b)$$

if S_1 , S_2 and a are assumed independent on x .

For $E=0$, these equations reduce to those in hydraulic theory of a varying flow of homogeneous fluid in an open channel. However, in the present case, R_i attains R_{in} for which $dR_i/dx=0$ in a short distance, afterwards remaining constant but $h(x)$ continues to increase according to $dh/dx = E(R_{in})$. From another analogy with the ordinary hydraulic flow, for $S_1 R_i > 1$ the flow is called tranquil in the sense that the velocity of long internal waves is greater than the flow velocity, and for $S_1 R_i < 1$ it is called shooting since the disturbances are unable to propagate upstream.

Two series of laboratory experiments: the spread of a surface

jet over a heavier fluid below and the flow of a heavy fluid down the sloping floor, show that E falls off rapidly as R_i increases and is probably negligible for R_i greater than about 0.8.

The laboratory results can be used for determining the velocity of flow of a larger scale from the values of A , i.e. a katabatic wind or the release of methane in a mine gallery. For the turbidity currents, A can be determined from the flow velocity, since in this case E becomes so small owing to small a that the relation $S_2 A U^{-3} \sin a = C$ is valid.

Ellison and Turner (1960a) made laboratory experiments on the behavior of a layer of dense salt solution introduced through a slit on the floor of a sloping rectangular pipe, in which the ventilating stream of fresh water flows with or against gravity. In the case of ventilation with gravity, the turbulence has little effect on the behavior of the system and mixing decreases over a wide range of the ventilating velocity as it increases. In the case of ventilation against gravity, two distinct cases arise according to the ventilating velocity; when it is large, all the salt solution is carried uphill, forming a layer whose thickness increases with distance (fully reversed flow), and when it is small, a part of the salt solution flows downhill and thus three regions of flow are distinguished. The rate of spread of the edge of the salt solution in the fully reversed flow is found to depend mainly on the slope a and on the pipe Richardson number $R_{ip} = D\Delta_d V^{-2} \cos a$ where D is the depth of the pipe, $\Delta_d = \frac{g(\rho_d - \rho_b)}{\rho_b}$ and ρ_d, ρ_b are the density of the fully mixed discharge and of the ambient flow, respectively.

In the range of R_{ip} from 0 to 0.505 the rate of spread decreases by a factor of about 3 at small slopes.

Ellison and Turner (1960b) computed the effect of the density difference on the velocity profile in a sloping pipe using the equation of motion integrated with depth z with an assumption that Δ and u are independent on x ,

$$(M - z) \frac{dp_R}{dx} + \int_z^D \sin a \cdot \Delta \, dz' + (\eta_M + \nu) \frac{du}{dz} = 0 \qquad (7.61)$$

where η_M is the eddy viscosity, M is defined by $(D-M) dp_R/dx = -\tau_R/\rho_b$ and p_R and τ_R are the pressure and the stress at the roof. This equation is further integrated with z , assuming that

$$\eta_M = \eta_0 \left(\frac{z}{D}\right)\left(1 - \frac{z}{D}\right) \qquad (7.62)$$

$$\Delta = \Delta_s \left(1 - z/h_A\right) \quad \text{for } 0 < z < h_A, = 0 \quad \text{for } h_A < z < D. \qquad (7.63)$$

where h_A is the height of the entrained salt layer. The results are compared with measurements obtained by timing streaks of dye at various levels in the pipe. These velocity profiles together with density profiles are used to determine the dependence of eddy viscosity η_M and eddy diffusivity of salt η_S on the local Richardson number $R_i = -(\partial\Delta/\partial z)(\partial u/\partial z)^{-2}$. It is found that η_S is much more greatly affected by the density gradient than η_M and that η_S / η_M is a decreasing function of R_i. The results agree with Ellison's (1957) prediction that

$$\eta_S / \eta_M \approx (1 - R_f / R_{fc})(1 - R_f)^{-2} \qquad (7.64)$$

where R_f is the flux Richardson number and R_{fc} is its critical value, at which $\eta_S = 0$. The value of R_{fc} determined is about 0.10. Also, the semi-empirical method is derived to give

the relation:

$$\eta_s / \bar{u} \, h_A \approx 0.22 \; dh_A / dx \qquad (7.65)$$

where $\bar{u} = D^{-1} \int_0^D u \, dz$. This relation and equation (7.62) give the ratio η_s / η_M in terms of the overall properties of the flow.

Ellison (1957) discussed the relations between turbulence intensities and mixing in a stratified fluid, deriving the eqautions for the mean square density fluctuation, the turbulent energy and the density flux from the continuity, the Navier-Stokes and heat conduction equations, respectively. When small terms are neglected, these equations become:

$$\overline{(\rho')^2} / (2T_1) + \overline{w\rho'} \; d\bar{\rho} / dz = 0 \qquad (7.66)$$

$$\overline{u_i^2} / (2T_2) + \overline{uw} \; dU / dz + \overline{w\rho'} g = 0 \qquad (7.67)$$

$$\overline{w\rho'} / (2T_3) + \overline{w^2} \; d\bar{\rho} / dz + \overline{\rho'^2} \, g / \bar{\rho} = 0 \qquad (7.68)$$

where T_1 , T_2 and T_3 are decay times respectively for $\overline{(\rho')^2}$, $\overline{u_i^2}$ and $\overline{w\rho'}$ which might be dissipated by turbulence in the absence of the producing effects.

Eliminating $\overline{(\rho')^2}$ from (7. 66) to (7. 68) and using

$$\eta_\rho = - \overline{\rho' w} \, (d\bar{\rho} / dz)^{-1} \; ; \; \eta_M = - \overline{uw} \, (dU / dz)^{-1} \qquad (7.69ab)$$

we have

$$\frac{\eta_\rho}{\eta_M} = \frac{\overline{u_i^2} \; \overline{w^2} \left[1 - R_f \left\{ 1 + (T_1 \overline{u_i^2})(T_2 \overline{w^2})^{-1} \right\} \right]}{2 u_*^4 \, (T_2 / T_3)(1 - R_f)^2} \qquad (7.70)$$

If we take $T_2 / T_3 \approx 1$, $\overline{u_i^2} / \overline{w^2} \approx 5.5$, the critical value of R_f at which η_ρ / η_M vanishes is 0.15.

If the length scale of turbulence L_M is defined as $T_2 (\overline{u_i^2})^{\frac{1}{2}}$, we have, from (7. 67) and (7. 11),

$$L_M = (\overline{u_i^2})^{3/2} \, u_*^{-3} \, L \, R_f (1 - R_f) \qquad (7.71)$$

The scale $L_\rho = \left[\overline{(\rho')^2} \right]^{\frac{1}{2}} (d\bar{\rho} / dz)^{-1}$ is defined as the vertical distance travelled by a particle which is lifted against

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the stability due to its kinetic energy. From (7.66) and (7.71) we have

$$L_\rho^2 = L_M^2 \frac{T_1 T_2}{g T_0^2} \frac{\overline{w^2}}{U_*^2} \frac{1 - R_f (R_{fc})^{-1}}{1 - R_f} \qquad (7.72)$$

which shows that in stable conditions L_ρ becomes much smaller than L_M.

There is no measurement which are precise enough to enable these relations to be checked. Ellison referred to the work of Proudman (1953), who realised currents and salinity in the Kattegat measured by Jacobsen (1913) and obtained the critical value R_{fc} around 0.25.

Ellison considered in this approach the transport process in a boundary layer transmitting constant shear stress and constant heat flux. But Townsend (1957) discussed the similar problem in a fluid far from restraining boundaries. He used the equation for change of r.m.s. of the temperature fluctuations θ

$$\overline{w\theta}\left(\frac{\partial T}{\partial z} + \frac{g}{C_p}\right) = \chi\, \overline{\theta \nabla^2 \theta} < -\frac{1}{3}\, \overline{\theta^2}\left(\overline{w^2}\right)^{1/2} L_\theta^{-1} \qquad (7.73)$$

and the equation for the kinetic energy of the velocity fluctuations.

$$\overline{uw}\,\frac{\partial U}{\partial z} - \frac{g}{T}\,\overline{w\theta} = \nu\, \overline{u_i \nabla^2 u_i} = -\left(\overline{w^2}\right)^{3/2} L_\varepsilon^{-1} \qquad (7.74)$$

for determining the flux Richardson number, where θ, u and w are the fluctuations of temperature, horizontal and vertical velocity, U and T are the mean horizontal velocity and temperature, and χ and ν are molecular diffusivity and molecular viscosity, respectively. The dissipation term is replaced by the term including the dissipation length scale on the basis of similarity hypothesis that turbulent dissipation of energy is proportional to 3/2 - power of the intensity. The replacement of the molecular conduction term by the term with

the turbulence length scale L_θ is to some extent formal, but it expresses the fact that the large-scale properties of turbulent transport are independent on molecular processes.

The eddy viscosity η_u and eddy diffusivity η_θ are respectively defined by

$$\eta_u = \overline{uw}\ (\overline{w^2})^{-1} \quad ; \quad \eta_\theta = |\overline{w\theta}|\ (\overline{w^2}\ \overline{\theta^2})^{-\frac{1}{2}} \qquad (7.75a,b)$$

Combining equations (7. 73) and (7. 74) leads to a quadratic equation for the turbulent heat transport, whose solution may be written in terms of R_f and R_i :

$$R_f = \frac{(g/T)\ \overline{\theta w}}{\overline{uw}\ (\partial U/\partial z)} = \frac{1}{2}\left[1 - \left(1 - \frac{1}{12}\frac{L_\theta}{L_\varepsilon}\left(\frac{\eta_\theta}{\eta_u}\right)^2 R_i\right)^{\frac{1}{2}}\right] \qquad (7.76)$$

where

$$R_i = \frac{g}{T}\left(\frac{\partial T}{\partial z} + \frac{g}{c_p}\right)\left(\frac{\partial U}{\partial z}\right)^{-2}$$

This equation indicates that the real values of the turbulent heat flux (or R_f) are only possible if

$$R_i < 12\left(\frac{\eta_u}{\eta_\theta}\right)^2 \frac{L_\varepsilon}{L_\theta} \qquad (7.77a)$$

This sets the limit to possible values of R_f

$$R_f < \frac{1}{2} \qquad (7.77b)$$

This limit is additional to the limit set by the energy equation alone $(R_f < 1)$ and it arises from the additional condition for the temperature fluctuations. The actual value of the limit of R_i was found to be about 0.08 from laboratory experiments as discussed in the earlier part of this chapter.

8. Empirical determination of vertical eddy viscosity in the ocean.

Many oceanographers developed different methods of estimating vertical eddy viscosity in the ocean after the idea of Austausch was introduced into oceanography about half a century ago. In general terms there are two categories in such methods. To the first category belong the widely used methods in which the equations of mean motion are applied to the observed distributions of velocity. Also, in many cases, the mean distributions of properties of water like temperature, salinity and oxygen were used in the transport equation to determine the eddy diffusivity. The second category includes the methods, in which the Reynolds stress are measured directly and the eddy viscosity is determined from the ratio of this stress to the shear of the mean flow. This approach has been taken recently by Bowen and Fairbairn(1956).

The methods of determining eddy viscosity from the mean velocity distributions can be divided into two types. One type is to use as a mean flow the currents measured with some averaging process like impeller type current meters or hydrographic cast for dynamic calculation of currents. In oceanography there are [illegible] few measurements which are repeated at the same place except in a shallow water. Therefore, in most cases the data obtained by one measurement were considered to represent the average condition. The second type is to use harmonic components of currents or concentrations obtained for periodic phenomena like tidal currents or daily and annual change of oceanographic elements.

The equations of momentum transport of the mean motion are given by:

$$\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u} + f\vec{k}\times\vec{u} = -\frac{1}{\rho}\nabla p + \frac{\partial}{\partial z}\left(\eta\frac{\partial\vec{u}}{\partial z}\right) \qquad (8\cdot1)$$

and the transport equation of concentration S is

$$\frac{\partial S}{\partial t} + \vec{u}\cdot\nabla S = \frac{\partial}{\partial z}\left(\eta_S\frac{\partial S}{\partial z}\right) \qquad (8\cdot2)$$

where z is the vertical coordinate, $\vec{u}$ is the velocity vector, $f\vec{k}$ is a rotation vector of the earth and η and η_S are the eddy viscosity and eddy diffusivity, respectively.

The first type of methods of determining eddy viscosity is essentially to integrate $(8\cdot1)$ or $(8\cdot2)$ with z. The result is

$$\left(\eta\frac{\partial\vec{u}}{\partial z}\right)_z - \left(\eta\frac{\partial\vec{u}}{\partial z}\right)_h = \int_h^z\left(\frac{\partial\vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u} + f\vec{k}\times\vec{u} + \frac{1}{\rho}\nabla p\right)dz \qquad (8\cdot3)$$

$$\left(\eta_S\frac{\partial S}{\partial z}\right)_z - \left(\eta_S\frac{\partial S}{\partial z}\right)_h = \int_h^z\left(\frac{\partial S}{\partial t} + \vec{u}\cdot\nabla S\right)dz \qquad (8\cdot4)$$

in which h means some reference level. These equations indicate that eddy viscosity or eddy diffusivity at a depth z can be determined when its value at the reference level h, in addition to the distributions of mean velocity and mean concentration, is known.

In many applications, several assumptions were made in order to simplify the computation. They are, for example, steady state ($\partial\vec{u}/\partial t = 0$ or $\partial S/\partial t = 0$), no vertical motion, or neglect of inertia terms ($\vec{u}\,\nabla\vec{u} = 0$).

The reference level is taken either at the sea surface or at the bottom. At the sea surface, the flux of momentum $(\eta\,\partial\vec{u}/\partial z)_h$

is equal to the wind stress which can be determined as a function of wind velocity as discussed before. Also, the flux of heat or salinity at the surface $(\eta_s \partial S/\partial z)_s$ is determined by heat balance between net radiation and sensible and latent heat transfer to the air or the difference of precipitation and evaporation, respectively.

When the bottom is taken as a reference level, the bottom stress $(\eta \, \partial \vec{u}/\partial z)_b$ is determined from application of the mixing length theory to the boundary layer close to the bottom. The flux of heat or salinity is usually considered to vanish there but the flux of suspended materials is finite and dependent on characteristics of bottom sediments and turbulent conditions near the bottom.

The examples of works on determining eddy viscosity or eddy diffusivity at various parts of the ocean by this method are summarized in the text books of Proudman (1953) and Defant (1961). One such which is often quoted is the study of Fjeldstad (1929), who determined the vertical distribution of eddy viscosity by using data of drift currents over the Siberian Shelf obtained by Sverdrup. The distribution is given by:

$$\eta = \eta_s \left(\frac{z + z_e}{H + z_e} \right)^{3/4} \qquad (8.5)$$

in which H is the depth, η_s is the surface eddy viscosity and z_e is a constant of a dimension of length. Fel'zenbaum (1956) modified this equation by applying the mixing length theory to the surface stress and obtained the equation:

$$\eta = 0.4 \, \gamma \, (\rho u_s)^{-1} W^2 H \, (1 - z/H)^{3/4} \qquad (8.6)$$

in which W and U_S is the wind speed and surface current speed, respectively and γ is the drag coefficient of wind.

The method was modified to determine the eddy diffusivity of properties which are not conservative. Ichiye (1952c) determined the eddy diffusivity of water temperature near the surface, including in the right side of equation (8.2) the term $Q_0 e^{-kz}$ which represents the absorption of radiation by water. Also, he (Ichiye, 1954) used transport equations of oxygen, phosphates and silicates with terms expressing the effects of consumption and production simultaneously with the equations for more or less conservative elements, temperature and salinity. Thus, he determined the vertical distributions of consumption-production function as well as the eddy diffusivity, assuming the same eddy diffusivity for each pair of elements. Later he (Ichiye, 1956) elaborated the method to compute the eddy diffusivity as a function of sigma-t instead of depth, because the correlation curves of non-conservative elements and sigma-t are almost identical for the same water mass, suggesting that the effects of horizontal advection and diffusion are eliminated. Okubo (1958) applied the idea of using simultaneously transport equations of temperature and oxygen to qualitative dissusion of oxygen compensation depth. Miyoshi and others (1955) discussed the vertical distributions of radioactivity of sea water in the equatorial Pacific polluted by nuclear bomb experiments at Bikini in 1954. Comparing theoretical distributions computed with an assumption of constant vertical eddy diffusivity with observed data, they estimated the diffusivity as 5 cm^2/sec. Although their theory is not rigorous, the proper analysis of the data

of distributions of such artificial radioactivity might give an excellent tool for determining eddy diffusivity as well as for trading a current.

The second method, estimation of eddy diffusivity by use of harmonic components was first derived by Fjeldstadt (1933) and was generalized by Ichiye (1952c), who included the terms of advection, convection and absorbtion of heat (or any source or sink). The latter started from the transport equation:

$$\frac{\partial\theta}{\partial t} + \vec{u}_h\cdot\nabla_h\theta = \frac{\partial}{\partial z}\left(\eta_\theta\frac{\partial\theta}{\partial z}\right) - \frac{\partial w\theta}{\partial z} + Q_\theta \quad (8\cdot7)$$

in which $\vec{u}_h$ and w are horizontal and vertical velocity, respectively, and Q_θ is the source function. It is assumed that $\theta = X e^{i(\varphi+\sigma t)}$, $Q_\theta = Q_0 e^{i(\varphi_1+\sigma t)}$, $\vec{u}_h\cdot\nabla_h\theta = Y e^{i\varphi_2} S$ (8·8)
in which X, Q_0, Y, φ, φ_1 and φ_2 are known functions of z. Unknown quantities to be determined, η_θ and w, are assumed to be independent on time. Substituting (8·8) into (8·7) and separati into real and imaginary-part

$$\frac{d}{dz}\left(\eta_\theta\frac{dX}{dz}\right) - \eta_\theta X\left(\frac{d\varphi}{dz}\right)^2 - \frac{d\,wX}{dz} + XY\cos\varphi_2$$
$$+ Q_0\cos(\varphi_1-\varphi) = 0 \qquad (8\cdot9)$$

$$\frac{1}{X}\frac{d}{dz}\left(\eta_\theta X^2\frac{d\varphi}{dz}\right) - wX\frac{d\varphi}{dz} + XY\sin\varphi_2 + Q_0\sin(\varphi_1-\varphi) = \sigma X$$
$$(8\cdot10)$$

The differential equations were transformed into difference equations by dividing the whole depth into small layer. Then, η_θ and w at each layer can be determined by solving a system of linear algebraic equation, when the values at the bottom are assumed to zero. The original equation of Fjeldstad is the second

equation (8·10), without terms containing W, Y and Q_0 .

Bowden and Fairbairn (1952) applied similar method determining bottom stress and eddy viscosity in tidal currents. They started from the $x-$ component of equations (8·1) without inertia terms like

$$\frac{\partial u}{\partial t} - f v = -g \frac{\partial \zeta}{\partial x} + \frac{\partial \tau}{\partial z} \qquad (8\cdot 11)$$

in which τ is the stress. Taking the average of this equation with depth first and then the average of two stations locating at $x = 0$ and $x = x_L$ respectively, they obtained:

$$[\partial \bar{u}/\partial t] - f[\bar{v}] = -g(\zeta_L - \zeta_0)/x_L - [(\tau_b - \tau_s)/H] \qquad (8\cdot 12)$$

in which the bars and brackets indicate the average with depth and of two stations, respectively and suffixes L, 0, b and s represent the values at $x = x_L$, $x = 0$, at the bottom and the surface, respectively.

The harmonic components of ζ, u, v and τ are substituted in equations (8·11) and (8·12) which yield two pairs of equations for amplitudes and phases of these components by separating coefficients of sine and cosine function of time. The amplitude and phase of the bottom stress were determined from the pair of equations obtained from (8·12), using observed tides and currents at two stations, under an assumption of vanishing surface stress. The stress at any depth was computed from the pair of equations resulting from (8·11), by using the bottom stress thus determined in addition to the data of tides and currents. The eddy viscosity η is then obtained from the relation:

$$\eta = \tau / |\partial u/\partial z| \qquad (8\cdot 13)$$

The values of η determined from the data in the Irish Sea range 10^2 to 5×10^2 cm^2/S.

Ichiye (1953b, 1955a) generalized the method of Bowen and Fairbairn. He used both components of the equations of motion (8·1) and assumed different values of eddy viscosity in x and y-directions. Also he considered the surface stresses which are determined by the wind stress. The eddy viscosity obtained from the data of tidal currents at seven stations in the Inland Sea of Japan ranged from 1 to 30 cm^2/s. The values determined from semi-diurnal components were larger than those determined from diurnal one and the former showed a tendency of decreasing with depth. Also, he assumed that the friction terms of equation (8·1) are proportional either to linear velocity or square of velocity with phase lags. He determined the proportional constants as well as the phase lags by using the same data as discussed above. The constant for linear proportionality ranged from 10^{-5} to 10^{-3} (sec^{-1}) and those for square proportionality ranged from 10^{-6} to 10^{-3} (cm^{-1}).

Such scattering in proportional constants determined may be not only due to inaccuracies in tidal currents measured by conventional instruments like Ekman-type current meter, but also are due to an inadequacy of definition of velocity which appears in the frictional terms. It is still to be studied what kind of velocity should be used in linear or quadratic relation of a frictional force in time-varying flows like tidal currents. Taylor (1919) used the quadratic law of friction $(= \gamma \rho \overline{U^2})$ similar to (3.21) in order to compute the dissipation of the tidal energy, which is for a full period T equal to $\int_0^T \rho \gamma |U(t)|^3 dt = 4 \gamma \rho (3\pi)^{-1} U^3$ where $U(t) = U \cos 2\pi (t/T)$.

Jacobsen (1927) introduced a method of determining vertical eddy diffusivity using two successive (in time or in the direction of mean current) T-S curves. If the depths of two T-S points which are intersected on the prior (in time) or the upstream curve by a tangent to the later or downstream curve are designated Z_1 and Z_2, the eddy diffusivity η_D can be computed by

$$\eta_D = (Z_1 - Z_2)^2 (8\Delta t)^{-1} \quad \text{or} \quad (Z_1 - Z_2)^2 \, U \, (8\Delta x)^{-1} \qquad (8.14)$$

where Δt is the time interval between two observations, and Δx is the distance between two stations along the direction of the mean current U. Stockman (1946) explained the validity of equation (8.14) from two Fickian diffusion equations for temperature and salinity and proposed a modified method.

The Reynold's stresses in the ocean was for the first time measured directly by Bowden and Fairbairn (1956), using an electromagnetic flow meter in the Irish Sea. The flow meter can measure two components of velocity between two pairs of electrodes which are mounted in a magnetic field produced by a circular coil. One of the advantages of this instrument is its capability to measure weak vertical current simultaneously with horizontal current with a lag of response less than one second. The measuring units were mounted on a tripod at the heights 50 cm to 175 cm from the bottom. Averages of several runs yield the Reynolds' stress $\overline{u'w'}$ of 4.1 dyne/cm^2 for mean bottom current U_b=40cm/sec and 2.1 dyne/cm^2 for U_b=32 cm/sec at the height 75 cm from the bottom. At the height of 150 cm, the Reynolds' stress if equal to 4.0 dyne/cm^2 for U_b = 50 cm/sec and

is equal to 1.4 dyne/cm^2 for U_b=50 cm/sec and is equal to stress k_b, which is determined from the relation

$$\overline{u' \, w'} \Big|_b = k_b \rho U_b^2 \qquad (8.15)$$

is equal to 2.4 x 10^{-3}. The roughness parameter z_0 can be

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(A note added later.) Several workers (Hesselberg, Ertel) on atmospheric turbulence had determined the Austausch coefficient using some statistical average of the fluctuations of meteorological elements long before the statistical theory of turbulence was developed. Ertel, (1930) assumed that a particle coming from a level $z=\xi$ to $z=0$ has the property

$$\theta = \theta_0 - \xi (\partial\bar{\theta}/\partial z) \qquad (a)$$

where θ_0 is the mean value at $z=0$

Let $\phi(w)$ be the distribution function for vertical velocity w which satisfied $\int_{-\infty}^{\infty} w \, \phi(w) dw = 0$. The average transport of θ through a horizontal surface is given by:

$$-\eta_\theta \frac{\partial\bar{\theta}}{\partial z} = \int_{-\infty}^{\infty} (\theta-\theta_0) w \phi(w) dw = -\left(\frac{\partial\bar{\theta}}{\partial z}\right) \int_{-\infty}^{\infty} \xi \phi(w) w \, dw \qquad (b)$$

Hesselberg (1929) postulated that $\xi(w) \approx a \; w \approx \frac{1}{4} T w$, where T is the period of the fluctuations of θ. Multiplying $\phi(w)$ with the square of (a) and the equation of conservation of θ, $\partial\theta/\partial t = -w \; (\partial\theta/\partial z)$ and integrating with w, we have

$$(\eta_\theta/a) \; (\partial\bar{\theta}/\partial z)^2 = \int_{-\infty}^{\infty} (\theta-\theta_0)^2 \phi(w) dw = \langle(\theta-\theta_0)^2\rangle \qquad (c)$$

$$(\eta_\theta/a^2) \; (\partial\bar{\theta}/\partial z)^2 = \int_{-\infty}^{\infty} (\partial\bar{\theta}/\partial t)^2 \phi(w) dw = \langle(\partial\bar{\theta}/\partial t)^2\rangle \qquad (d)$$

Eliminating a from (c) and (d), we have

$$\eta_\theta = \langle(\theta-\theta_0)^2\rangle^{\frac{1}{2}} \langle(\partial\bar{\theta}/\partial t)^2\rangle^{\frac{1}{2}} \quad (\partial\bar{\theta}/\partial z)^{-2} \qquad (e)$$

where the terms of the right side can be determined from the observation. Ertel (1932) generalized the relation (b) to the three dimensional case and derived the Austausch tensor.

determined from bottom stresses $\overline{u'w'}_b$ and mean velocity U at several levels using the logarithmic formula of the mean current. The values of z_0 thus determined are scattered from 0.02 to 0.68 mm with the median of 0.21 mm. The authors mentioned that the reason of such scattering lies in the errors in estimation of the mean velocity. However, there is a possibility that the mixing length hypothesis is not valid in such a flow as tidal currents which change with time.

Nan'niti (1956) measured fluctuations of horizontal velocity at a shallow water of 10 meter depth using his photoelectric current meter. Since his instrument could measure the currents only at one depth, he tried to determine autocorrelations and then to estimate the turbulence scale. He also discussed the validity of measurements with a finite duration using Ogura's theory (1952). However, his instrument does not seem to have enough sensitivity to measure rapid and small fluctuations of velocity occurring in turbulence of small scale.

Recently Kolensnikov and others (1958) constructed a highly sophisticated instrument "turbulimeter" for measuring small scale turbulence in the ocean. This instrument can record the mean values and fluctuations of temperature and horizontal velocity at two depths, the distance of which varies from 0.5 to 2 m, and the fluctuations of vertical velocity at the top depth. The temperature sensors are thermistors, which can measure temperature fluctuations up to 10^{-3}°C with a time lag less than 0.08 sec. The current sensors are of hot-wire anemometer type and can measure the fluctuations up to 0.1 mm/sec. Salinity is simultaneously measured with a bathytemperature-salinograph. This type of instru-

ments were used in the Black and Caspian Seas, at the drifting ice stations of the Polar Sea, in Lake Baikal (Speranskaya, 1959) and aboard the "Ob" in the Antarctic Ocean (Panteleev, 1959).

Kolensnikov (1959, 1960) discussed the data separately for the homogeneous layers $(\partial\bar{\xi}/\partial z \approx 0)$ and for the stably stratified layers $(\partial\bar{\xi}/\partial z > 0)$. As an example for the neutrally stratified case, he discussed the data obtained in the sub-ice layer at the Polar floating station. In fig. 6 a and b, are plotted the vertical distributions of measured values of $\bar{u}$, $\sigma_u = [\overline{(u')^2}]^{\frac{1}{2}}\bar{u}^{-1}$, $\sigma_w = [\overline{(w')^2}]^{\frac{1}{2}}\bar{u}^{-1}$, eddy viscosity $\eta\left(= -\overline{u'w'}/(\partial\bar{u}/\partial z)\right)$ and tangential stress τ $(= -\rho\overline{u'w'})$, respectively. Near the lower layer of the floating ice the anisotropy of turbulence is seen from the fact that $\sigma_u > \sigma_w$. Further he used the data to prove Kolmogorov's hypothesis on structure of locally isotropic turbulence, which will be discussed in the next chapter.

The data at a station in the Antarctic Ocean (st. 332) and at Lake Baikal are shown in Fig. 7 and 8 as examples of turbulence in the stably stratified sea. The values of eddy viscosity η and eddy diffusivity η_θ are calculated from the formula

$$\eta = -\overline{u'w'}\,(\partial\bar{u}/\partial z)^{-1}, \quad \eta_\theta = -\overline{\theta'w'}\,(\partial\bar{\theta}/\partial z)^{-1} \qquad (8\cdot16a,b)$$

These quantities show similar change with depth, become larger in the layer of maximum horizontal velocity, but are almost inversely proportional to Richardson number. The ratio η/η_θ is larger for larger values of R_i and sometimes it reaches 50.

The eddy viscosity was determined by two different formulas, using the data presented in Fig. 7. One is Prandtl's formula $\eta = l^2\left|\frac{d\bar{u}}{dz}\right|$. The other formula is:

$$\eta = a_E \, l \left(\overline{u'^2} + \overline{w'^2} \right)^{\frac{1}{2}} \qquad (8.17)$$

where a_E is a constant close to one. In these formula l was termed "turbulence scale" by Kolesnikov instead of mixing length. This length can be determined by three formulas:

$$l = \overline{u} \int_0^\infty R(t)\,dt \quad ; \quad l = \left[\overline{u'^2} + \overline{w'^2} \right]^{\frac{1}{2}} (d\overline{u}/dz)^{-1}$$

$$l = \left(\overline{\theta'^2} \right)^{\frac{1}{2}} (d\overline{\theta}/dz)^{-1} \qquad (8.18\ a, b, c)$$

in which $R(t)$ is the autocorrelation of horizontal velocity. The values of l and η determined at several depths are plotted in Fig. 7 a and b, respectively. The instruments, which seem to be quite useful to measure oceanic turbulence were recently developed by Klaus (1960) and Geul (1962). The former constructed a mast consisting of several ten meter tubes of noncorrodible material in order to measure temperature, currents and the angle of the mast up to the water 200 m deep. Temperature sensors are platinum resistance thermometers and current meters are of pendulum type which measures the deflected angle of an inverted pendulum for speed. Recording units are kept in an iron case anchoring the mast and are operated by batteries for several days. This system was originally developed for measuring internal waves. However, it might be used successfully for measuring turbulence of intermediate scales both in vertical and horizontal directions, since it can be left unattended for a while though sensitivity of the current meters are not good. In fact, Klaus and Magaard (1962) reported the spectrum of temperatures and current fluctuations measured with this system in the Baltic Sea, although they could not find

an evidence of spectrum predicted by statistical theory of turbuelnce, but obtain discrete spectrum of internal oscillations.

Gaul also started to measure continuously temperature and currents at several depths on the northern coast of the Gulf of Mexico using buoys and telemetering system. He used the thermistors for temperature sensors and the Servonius rotor current meters. These instruments are highly sensitive and the whole system can be left unattended almost for a month. It is expected that the system will provide the data of whole ranges of turbulence from small to medium scales.

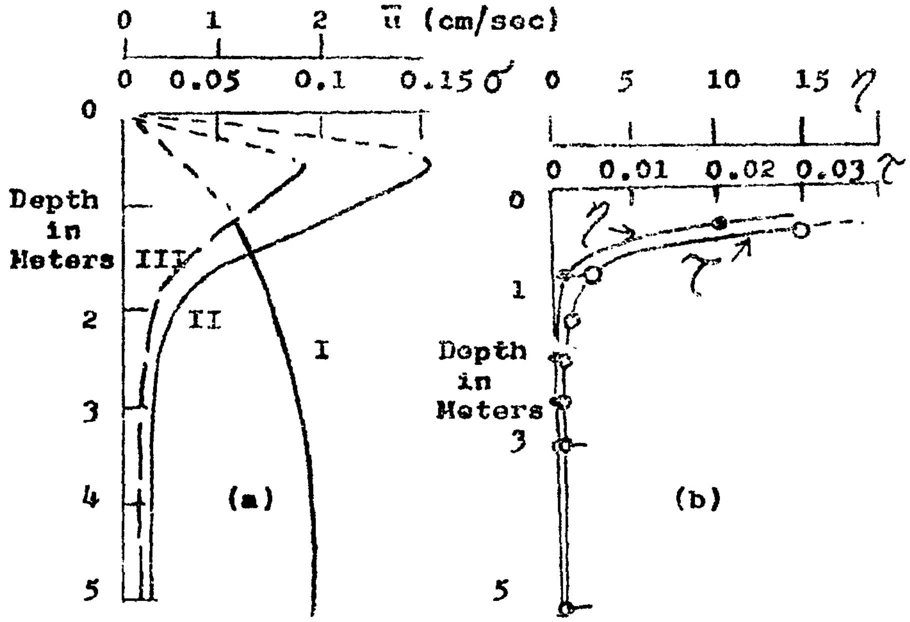


Fig. 6. Vertical distributions of velocity and velocity fluctuations (a) and eddy viscosity η and shearing stress τ (b). (Kolesnikov, 1960) (Curve I: $\bar{u}(z)$, II: $\sigma_u = (\overline{u'^2})^{\frac{1}{2}}/u(z)$ III: $\sigma_w = (\overline{w'^2})^{\frac{1}{2}}/u(z)$. Measurements at the Polar drifting station "North Pole-4".)

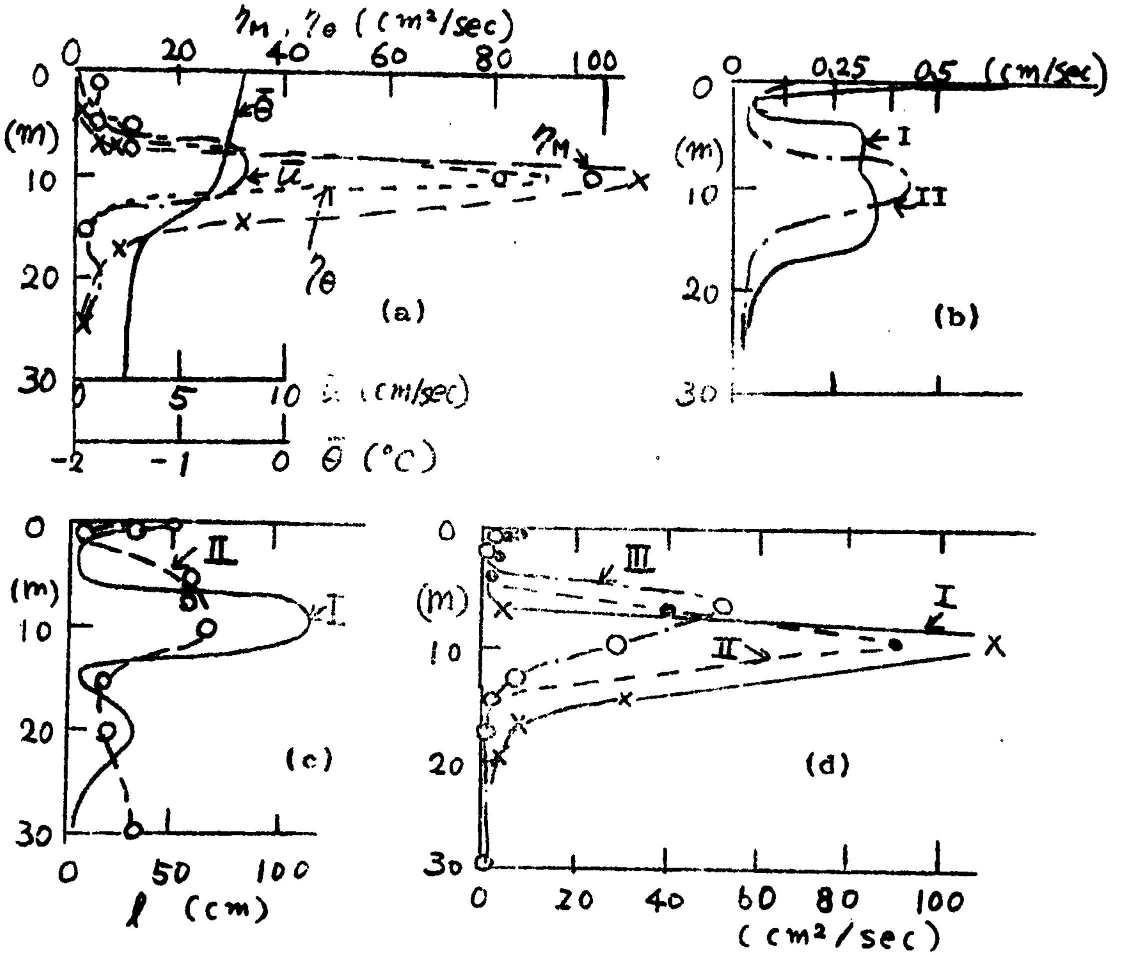


Fig. 7. Vertical distributions of various elements related to turbulence, based on measurement in the Antarctic Ocean station. (Kolesnikov, 1960) (a) —— $\bar{\theta}$, ------ $\bar{u}$, ···o·· η_θ, —x— η_M

(b) Curve I $\left[\overline{(\theta')^2}\right]^{\frac{1}{2}}$, II $\left[\overline{(w')^2}\right]^{\frac{1}{2}}$

(c) Curve I $l = \bar{u}\int_0^{\infty} R_t(\xi)\,d\xi$, II $l = \left(\overline{\theta'^2}\right)^{\frac{1}{2}} / (d\bar{\theta}/dz)$.

(d) Curve I $\eta_M = -\overline{u'w'}/(\partial\bar{u}/\partial z)$, II $\eta_M = a_z l\left[\overline{(u')^2} + \overline{(w')^2}\right]^{\frac{1}{2}}$,

III $\eta_M = l^2(\partial\bar{u}/\partial z)$.

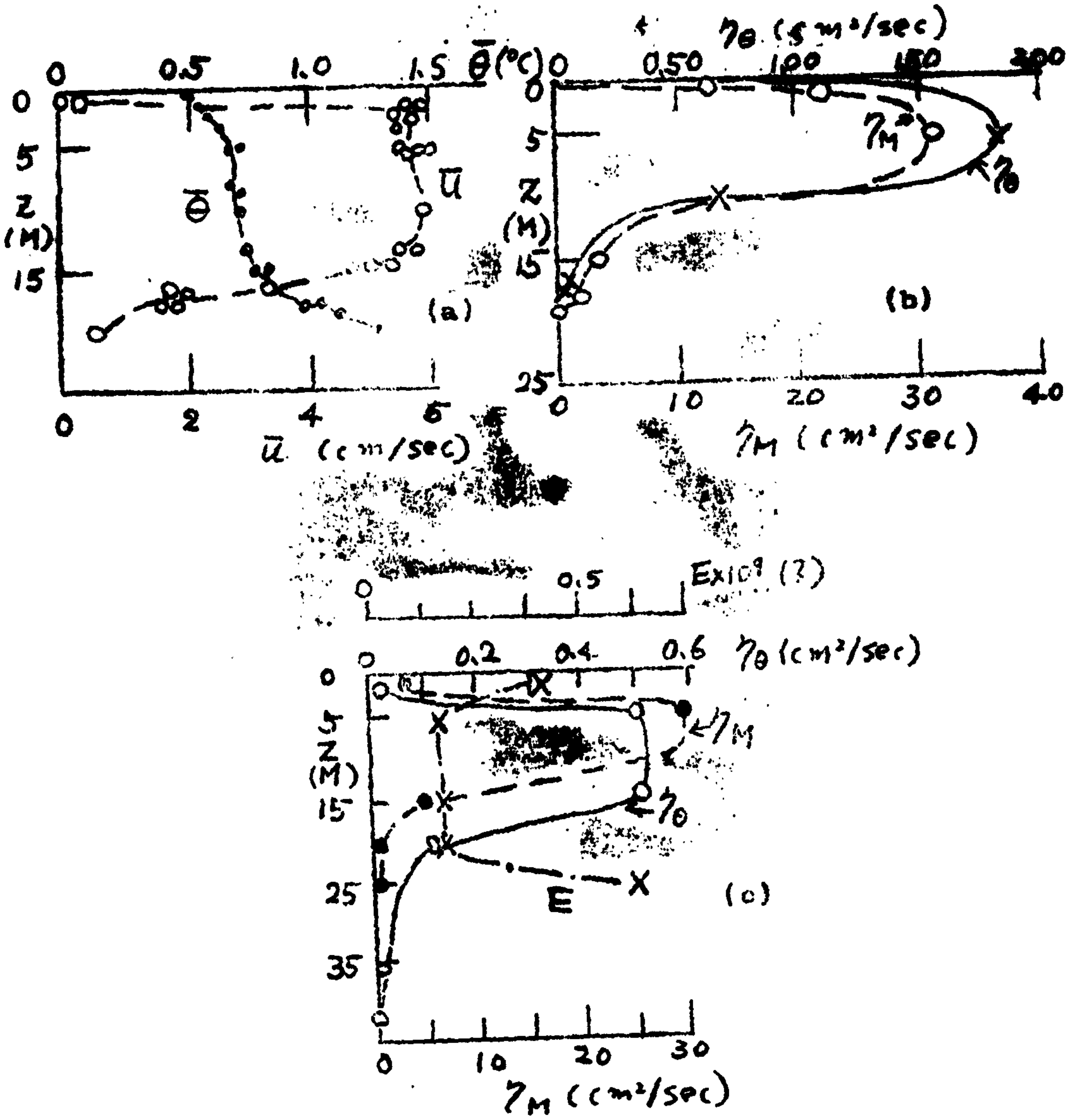


Fig. 8. Vertical distributions of various elements related to turbulence, based on the measurements in Lake Baikal. (Kolesnikov, 1960)

Pritchard (1952, 1956) and Pritchard and Kent (1956) discussed extensively the method of determining Reynolds stress in a coastal plain estuary, using averaged values of velocities, pressures and salinities. The equations of motion averaged with time can be written in a tensorial form as:

$$\frac{\partial \overline{u_i}}{\partial t} + \frac{\partial \overline{u_i}\,\overline{u_j}}{\partial x_j} - f\,\epsilon_{ij3}\,\overline{u_j} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} - \frac{\partial \overline{u_i' u_j'}}{\partial x_j}$$

$$(i = 1, 2\ ;\ j = 1, 2, 3) \qquad (8.19)$$

in which the bar and the prime indicate the averaged and instantaneous values, respectively, indices 1 and 2 represent horizontal coordinates, f is a Coriolis' parameter and ϵ_{ij3} is a component of a cyclic tensor.

For an elongated estuary, averaging of these equation (8.19), over the lateral direction yields much simpler form. If x- , y- and z- coordinates are taken along the longitudinal, lateral and vertical direction, respectively, equations of motion in x - and y - direction become:

$$\left\langle \frac{\partial \overline{u}.\overline{u}}{\partial x} \right\rangle + \left\langle \frac{\partial \overline{w}\,\overline{u}}{\partial z} \right\rangle = -\frac{1}{\rho}\left\langle \frac{\partial \overline{p}}{\partial x} \right\rangle - \frac{1}{b}\,\frac{\partial\, b \langle \overline{u'w'} \rangle}{\partial z} - \left\langle \frac{\overline{u'u'}}{\partial x} \right\rangle \qquad (8.20)$$

$$0 = -\frac{1}{\rho}\left(\overline{p}_r - \overline{p}_l\right) + f \langle \overline{u} \rangle - \frac{1}{b}\,\frac{\partial\, b \langle \overline{v'w'} \rangle}{\partial z} \qquad (8.21)$$

respectively, where $\langle\ \rangle$ means averaging over the y- direction,

$b(y,x)$ is the width of the bay and $\bar{p}_r$ and $\bar{p}_l$ are the values of $\bar{p}$ on each bank. In this direction, it is assumed that

$$\langle \bar{v} \rangle = 0 \quad , \quad \langle \overline{u'v'} \rangle = \langle \overline{v'w'} \rangle = 0$$

due to the elongated shape. Also, the term $\partial \overline{u'u'}/\partial x$ is considered negligible, since the corresponding term of salinity transport $\langle \overline{u'S'} \rangle$ was found negligible from the observations. The integration of (8.21) yields:

$$\langle \overline{u'w'} \rangle = -\frac{1}{b}\int_0^z \langle b\left(\frac{\partial(\bar{u})^2}{\partial x} + \frac{\partial \bar{u}\bar{w}}{\partial z} + \frac{1}{\rho}\frac{\partial \bar{p}}{\partial x}\right) \rangle dz \qquad (8{\cdot}22)$$

in which the condition that $\langle \overline{u'w'} \rangle = 0$ at the surface $z = 0$ is used. In this equation $\bar{w}$ can be computed from observed values of $\bar{u}$ and b using the equation of continuity, although in practice such calculation might introduce sources of errors. The pressure gradient is computed from the hydrostatic equation

$$\partial \bar{p}/\partial x = g\,\partial\left(\int_{z_s}^{z} \bar{\rho}\, dz\right)/\partial x \qquad (8{\cdot}23)$$

in which z_s is the depth of the surface. The density $\bar{\rho}$ is determined from hydrographic data and z_s from the condition that $\langle \overline{u'w'} \rangle$ should vanish at the bottom. Then the equation (8.22) yields the values of $\langle \overline{v'w'} \rangle$. The salinity transport by vertical turbulence $\langle \overline{S'w'} \rangle$ is determined simply from

$$\langle \bar{u}\,(\partial \bar{S}/\partial x) \rangle + \frac{1}{b}\,\frac{\partial\, b\,\langle \overline{S'w'} \rangle}{\partial z} = 0 \qquad (8{\cdot}24)$$

because observations indicate that neglected terms are smaller at least by two magnitudes.

Further Pritchard divided the average currents into two parts: $\overline{u_i}$, the mean velocity over one or more tidal cycles and U_i , the tidal contribution. The equation of motion in x_2 (lateral) direction averaged over one or more tidal cycles becomes:

$$\partial \bar{u}_j \overline{U_2}/\partial x_j + \langle \partial(U_j U_2)/\partial x_j \rangle_p = -\langle \rho^{-1}\partial \bar{p}/\partial x_2 \rangle_p + f\,\overline{U_1} - \langle \partial(U_j' U_2')/\partial x_j \rangle_p \qquad (8{\cdot}25)$$

in which $\langle\ \rangle_T$ means averaging as specified. The direction was so chosen that $\overline{u}_2 = U_2 = 0$. Thus,

$$0 = -\left\langle \frac{1}{\rho}\frac{\partial p}{\partial x_2}\right\rangle_T + f\,\overline{u}_1 - \left\langle \frac{\partial u_j' u_2'}{\partial x_j}\right\rangle_T \qquad (8.26)$$

Now, from measurements at some estuaires, $\left\langle \frac{\partial u_1' u_2'}{\partial x_1}\right\rangle_T$ and $\left\langle \frac{\partial (u_2')^2}{\partial x_2}\right\rangle_T$ seems to be very small, so equation (8.26) is written

$$\left\langle \frac{\partial u_2' u_3'}{\partial x_3}\right\rangle_T = f\,\overline{u}_1 - \left\langle \frac{1}{\rho}\frac{\partial p}{\partial x_2}\right\rangle_T \qquad (8.27)$$

Thus, Reynolds stress $\langle u_2' u_3'\rangle_T$ can be determined, using measurable quantities $\overline{u}$ and $\left\langle \frac{1}{\rho}\frac{\partial p}{\partial x_2}\right\rangle_T$

However, Stewart (1957) pointed out that such procedure might lead to quite erroneous Reynolds stress. Although $\overline{u}_2 \approx 0$, $U_2 \approx 0$, the inertia terms, $\frac{\partial \overline{u}_j \overline{u}_2}{\partial x_j}$ and $\frac{\partial U_j U_2}{\partial x_j}$ are in general not negligible. In the actual case treated by Pritchard (1956), the predominant part of inertia terms are:

$$\frac{\partial \overline{u}_2}{\partial x_1} \sim \frac{\overline{U}_1}{R}\ , \quad \frac{\partial U_2}{\partial x_1} \sim \frac{U_1}{R} \qquad (8.28\ a,b)$$

where R is the radius of curvature of the flow. These terms are of the same order of magnitude as the Coriolis' term $f\,\overline{u}_1$.

9. Spectral Analysis of Oceanic Turbulence

Since the turbulence is statistical phenomenon, it is no wonder that some workers on turbulence have used the approaches which are more or less pertinent to statistical mechanics. The pioneers of turbulence theory thus assumed the analogy, with a classical statistical mechanics of gas molecules, as described before. However, Taylor (1935) for the first time treated turbulence statistically with a clear understanding of the fact that turbulent velocity of the fluid is a random continuous function of position and time, in contrast to older theories based on analogies with the discontinuous collisions among the discrete entities.

The essential part of statistical theory of turbulence is an introduction of the covariance between the velocity components at two separated points and an application of a theory of Fourier integral to derivation of spectral density from the covariance. The covariance or velocity correlation is defined by:

$$R_{ij}(\vec{x};\vec{r}) = \overline{u_i(\vec{x})\,u_j(\vec{x}+\vec{r})} \qquad (9.1)$$

where $u_i(\vec{x})$ is an instantaneous value of the i components of the velocity fluctuation at the point $\vec{x}$. The average is taken over all realization of the flow field or, in case of stationary flow, over the large time interval. Incompressibility and interchangiability of u_i and u_j impose restrictions on the form of R_{ij} such as:

$$\frac{\partial}{\partial r_j} R_{ij}(\vec{x};\vec{r}) = 0 \qquad (9.2)$$

$$R_{ij}(\vec{x};\vec{r}) = R_{ji}(\vec{x}+\vec{r};-\vec{r}) \qquad (9.3)$$

When the turbulence is homogeneous, correlation function is independent on coordinates $\vec{x}$. Further, when the turbulence is both homogeneous and isotropic, more limitations of the form of R'_{ij} are derived. For instance, $R_{ii}(0)$ or the mean square values of three velocity components are equal to each other. Also, in the homogeneous and isotropic turbulence, the correlation function R_{ij} is represented by correlation tensors such as:

$$R_{ij} = \overline{u^2}\left[\frac{f(r)-g(r)}{r^2}\, r_i r_j + g(r)\,\delta_{ij}\right] \qquad (9.4)$$

in which $\overline{u^2}$ is the mean square value of velocity and δ_{ij} is the Kronecker delta. The functions $f(r)$ and $g(r)$ are defined as longitudinal and transverse correlation coefficient, respectively. In other words, if u_r and u'_r are velocity components parallel to the vector $\vec{r}$ joining two points P and P^1 at which the velocities are considered, $f(r)$ is defined by:

$$\overline{u_r u'_r} = \overline{u^2}\, f(r) \qquad (9.5)$$

The function $g(r)$ is defined by:

$$\overline{u_t u'_t} = \overline{u^2}\, g(r) \qquad (9.6)$$

in which u_t and u'_t are velocity components at P and P^1 perpendicular to $\vec{r}$.

The idea of spectral analysis which has long been used in electromagnetic theory of heat and light was introduced by Taylor (1935) into the study of turbulence. The idea is particularly

powerful for studying theory of isotropic turbulence. The efforts in the recent decades are directed to derive from the Navier-Stokes equations the functional form of the spectral density of turbulence. The most general form of the spectrum function is defined by using the three-dimensional Fourier transform of such as:

$$R_{ij}(\vec{x};\vec{r}) = \int_{-\infty}^{\infty} \Phi_{ij}(\vec{x};\vec{k}) e^{i\vec{k}\cdot\vec{r}} \, dV(\vec{k}) \qquad (9.7)$$

where $dV(\vec{k})$ is the element of volume about $\vec{k}$.

The integrated spectrum function is defined by:

$$E_{ij}(k) = \int \Phi_{ij}(\vec{k}) \, dS(\vec{k}) \qquad (9.8)$$

where the integration is over the spherical surface of radius k. Owing to incompressibility, there is a relation such as

$$k_j \Phi_{ij}(\vec{x};\vec{k}) = 0 \qquad (9.10)$$

A theoretical approach to the spectral theory of turbulence is to derive the spectral function from the transformation of Navier-Stokes equation. The triple correlation function which is produced by the nonlinear terms of Navier-Stokes equations can not be related deductively to the energy spectral function and this approach is so far only partly successful under restricted conditions of stationary, homogeneous and isotropic turbulence

for a certain range of scales. The empirical approach is widely used in describing measurement of wind tunnel turbulence, particularly by use of hot wire techniques, which give directly the data on correlation functions. The correlation functions are transformed into spectral functions by the Fourier inverse integral of (9.7). Many aerodynamicists have concentrated their efforts to applying hot wire techniques to the study of the turbulent structure of various flow fields and quite a number of separate pieces of information on this subject have been accumulated. However, the statisical approach to the turbulence are not fully exploited in the field of oceanography, owing to complications arising from anisotropy and inhomogeneity of ocean flow in the theoretical approach and difficulties in measuring fluctuations of the flow in the sea in the empirical approach.

Works by Bowden (1962), Kolesnikov (1960) and Grant et al(1961) are the first step toward filling the gap between the development in research on turbulence in wind tunnels and slow progress in research on oceanic turbulence. Bowden (1962) obtained the spectra of fluctuations of u, v and w, which represent the component in the direction of and perpendicular to the mean horizontal velocity and vertical current, respectively. His procedure is essentially similar to the one used by aerodynamicists in wind-tunnel experiments, except instruments used. He measured velocity correlations with the electroflow meter and derived the spectral function using the following relation:

$$F(n) = 4\frac{\overline{u^2}}{U}\int_0^\infty f(r)\cos\frac{2\pi n r}{U}\,dr = \frac{4}{U}\int_0^\infty R_{11}\cos\frac{2\pi n r}{U}\,dr \qquad (9.11)$$

in which the correlations are determined under the assumption that the velocity fluctuations are carried away by the mean flow unchanged, as postulated by Taylor (1935).

The results of Bowden's measurements are different from those obtained in both wind tunnels and from the theory of isotropic turbulence. First, the mean square velocities $\overline{u^2}$, $\overline{v^2}$ and $\overline{w^2}$ are not equal to each other. The averaged ratios $[\overline{v^2}/\overline{u^2}]^{\frac{1}{2}}$ and $[\overline{w^2}/\overline{u^2}]^{\frac{1}{2}}$ are equal to 0.51 0.76, respectively Correlation functions $f(r)$ and $g(r)$ derived from $\overline{uu'}$ and $\overline{ww'}$ respectively, do not satisfy the relation:

$$g(r) = f(r) + \frac{r}{2}\,\frac{df(r)}{dr} \qquad (9\cdot12)$$

which can be derived from equation of continuity under the assumption of isotropicity. Also, one dimensional spectra of three components of velocity u, v and w are different from those expected from the theory of isotropic turbulence. This is expected because the measurements of turbulent velocities were done close to the bottom at the heights of 50, 75, 125, and 150 cm from the bottom where the assumption of isotropicity is not valid.

Fig. shows the spectra of three components u, v, and w and the spectra of $\overline{uw}$ at four heights from the bottom. For a range of wave number 0.02 to 0.2 m^{-1}, the spectra of $\overline{uw}$ reach more than one third of those of u or w and this is another indication of anisotropy of turbulence. The $\overline{uw}$ spectra show very little contributions from the fluctuations with wave number higher than 0.6/z. This limiting wave number the turbulence is isotropic was the same as found by Priestley, (1959)

in his analysis of the atmospheric boundary layer obtained by Panofsky (1953) and others.

Following Kolmogorov (1941), Russian workers on turbulence have often used structure functions defined by:

$$D_{ik}(\vec{r}) = \overline{(u_i - u_i')(u_k - u_k')} \qquad (9.13)$$

instead of correlation functions (Tatarski, 1960, chapter 2). Here i,k = 1, 2, 3, the u_i are the components with respect to the x, y, z axes of the velocity vector at the point $P\ (= \vec{r}_1)$ and the u_i' are the components of the velocity at the point $P'\ (= \vec{r}_1 + \vec{r})$.

From the assumption of local isotropy of velocity, $D_{ik}(\vec{r})$ has the form

$$D_{ik}(\vec{r}) = [D_{rr}(r) - D_{tt}(r)]\, n_i n_k + D_{tt}\,\delta_{ik} \qquad (9.14)$$

where δ_{ik} is the Kronecker's delta and the n_i are the components of the unit vector along $\vec{r}$. Suffixes r and r in the definitions $D_{rr}(r) = \overline{(u_r - u'_r)^2}$, $D_{tt}(r) = \overline{(u_t - u'_t)^2}$ represent the projection of the velocity along the direction of $\vec{r}$ and along some direction perpendicular to $\vec{r}$, respectively. The condition that $\operatorname{div}\vec{u} = 0$ yields the relation:

$$D_{tt} = \frac{1}{2r}\frac{d}{dr}(r^2 D_{rr}) \qquad (9.15)$$

If we consider that a velocity fluctuation u'_l occurs in a region of scale l, characteristic time required for the growth of such fluctuation becomes l / u'_l. The energy transfer from the mean flow to the fluctuation per unit time is equal to $(u'_l)^2 (u'_l / l) = (u'_l)^3 l^{-1}$. The process of energy transfer in the fluid is as follows:

First, the mean flow with characteristic velocity U and scale L becomes unstable when its Reynolds number [illegible] exceeds a critical value and transfers its energy to the fluctuation (or eddies) created by this instability. The fluctuation itself becomes again unstable when the Reynolds number $(\nu u'/\ell')$ exceeds a critical value and transfers its energy to the second order fluctuation. This transfer process goes down to the smallest disturbance with a scale ℓ_0 and characteristic velocity u_0 and is finally converted into heat by molecular viscosity. The rate of such dissipation ε is of order $\nu u_0^2/\ell_0^2$. On the other hand, the rate of supply of energy from the fluctuation larger by one order to this smallest one is $u_0^3/\ell_0 (\sim \varepsilon)$. Thus, from these two formulas for ε, we can obtain

$$\ell_0 \sim (\nu^3/\varepsilon)^{1/4} , \quad u_0 \sim (\nu \varepsilon)^{1/4} \qquad (9.16)$$

Let $\vec{r}$ in equation (9.13) be much larger than ℓ_0 and much smaller than L, (the scale of the largest, anisotropic eddies). Then the velocity difference at the two points is mainly due to eddies with dimensions comparable to r. The only parameter which characterizes such eddies is ε and r. Thus, $D_{rr}(r)$ is a function of r and ε only.

From dimension grounds.

$$D_{rr}(r) = C \, (\varepsilon r)^{2/3} \qquad (\ell_0 \ll r \ll L) \qquad (9.17)$$

where C is a dimensionless contant of about [illegible]. From the relation (9.15), the quantity $D_{tt}(r)$ is given by

$$D_{tt}(r) = \frac{4}{3} C \, (\varepsilon r)^{2/3} \qquad (\ell_0 \ll r \ll L) \qquad (9.18)$$

Kolesnikov (1960) used the data obtained at the Antarctic Ocean (Fig.) to prove the relation (9.17). He determined $D_{nn}(r)$ from the measurement of horizontal velocity at one depth under an assumption that $r = \overline{u}t$, where $\overline{u}$ is the mean current. The $D_{nn}(r)$ is plotted against r for different values of mean velocity in Fig.

The range of scale of turbulence in which the rate of energy transfer from eddies of larger scale to eddies of smaller scale is constant is called inertia subrange. Kolmogorov's hypothesis consists of three parts: the first part is that the local characteristics of turbulence in the inertial subrange are isotropic at large Reynolds numbers, the second is that the energy supplied from a range of larger turbulence to the inertial subrange is completely transfered to a range of smaller turbulence with a constant rate, and the third is that the energy thus transfered to the range of smaller turbulence (dissipation range) is mainly dissipated by viscous forces into heat.

In the inertial subrange, the effect of viscosity can be neglected. Therefore, on the dimensional grounds the one-dimensional spectral density is given by:

$$E(k) = C_E \, \varepsilon^{\frac{2}{3}} \, k^{-\frac{5}{3}} \qquad (9\cdot19)$$

in which k is the wave number and C_E is an universal constant. The function $E(k)$ also can be expressed by:

$$E(k) = L_a^{-1} \left\{ \int_0^{L_a} u(x) e^{ikx} dx \right\} \left\{ \int_0^{L_a} u(x) e^{-ikx} dx \right\} \qquad (9\cdot20)$$

in which L_a is a length scale much larger than k^{-1}. This form can

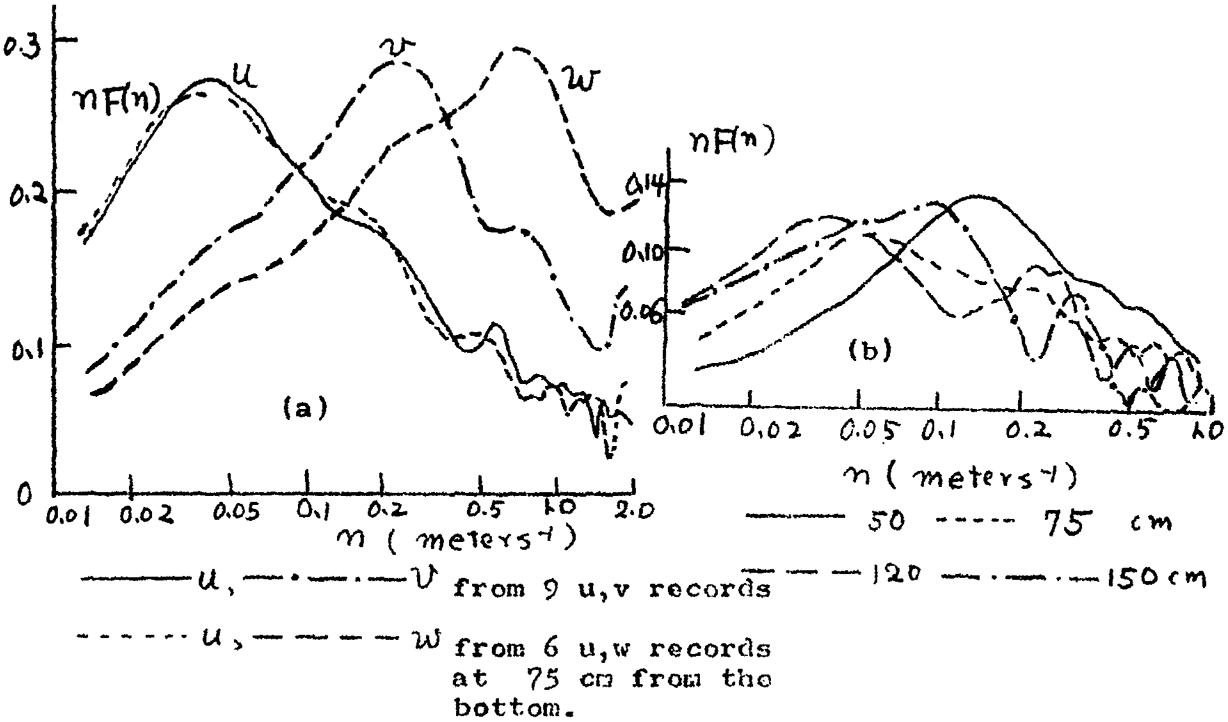


Fig. 9. Spectra of u v, and w (a) and uw(b) in the tidal currents.(Bowden,1962)

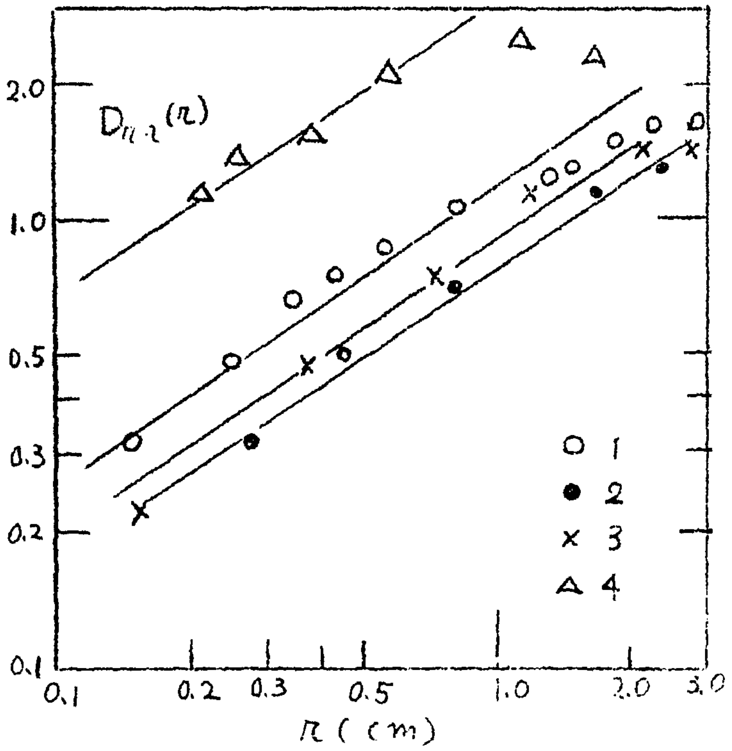


Fig. 10. The structure functions at four depths from the surface, based on measurements in the Polar sea.(Kolesnikov,1960)

1. 2 m, $[\overline{(u')^2}]^{\frac{1}{2}} = 1.9$ cm/sec
2. 5 m, $[\overline{(u')^2}]^{\frac{1}{2}} = 2.0$ cm/sec
3. 2.5 m, $[\overline{(u')^2}]^{\frac{1}{2}} = 1.8$ cm/sec
4. 1.5 m, $[\overline{(u')^2}]^{\frac{1}{2}} = 1.4$ cm/sec

be used to compute $E(k)$ if turbulent velocity component $u(x)$ is measured.

In the range of wave numbers including both inertia subrange and dissipation range, the function $E(k)$ is given by:

$$E(k) = \varepsilon^{1/4} \nu^{5/4} F(k/k_s) \qquad (9\cdot21)$$

in which $k_s = (\varepsilon \nu^{-3})^{1/4}$ and $F(k/k_s)$ is an universal function. On the basis of Heisenberg's assumption, Reid (1960) derived analytically the form of $F(k/k_s)$:

$$F(k/k_s) = C_F \, (k/k_s)^{-\frac{5}{3}} \qquad \text{for } k < \beta_R k_s$$

$$F(k/k_s) = C_F \, (k/k_s)^{-7} \qquad \text{for } k > \beta_R k_s$$

$$(9\cdot22)$$

in which β_R is a numerical constant.

The energy dissipation rate ε is given by:

$$\varepsilon = 15 \nu \int_0^{\infty} k^2 E(k) \, dk \qquad (9\cdot23)$$

which can be determined if $E(k)$ is obtained from measurements.

Grant, Stewart and Moillet (1962) constructed a hot-film current meter in order to determine the spectral density of oceanic turbulence and to test the Kolmogorov's hypothesis. The current meter which was mounted to a towed body has a frequency response from D.C. to 700 C.P.S. with a noise level of 10^{-2} cm/sec in the frequency band 1 to 700 C.P.S.

The measurements were done in Discovery Passage near Vancouver at the tidal currents of maximum speed of about three knots. They determined the energy spectrum $E(k)$ and

dissipation spectrum $k^2E(k)$ from the measurements at some 15 meters deep in a region of very intense turbulence, but very small surface waves. When the spectrum is plotted against the wave number the energy spectrum is well separated from the dissipation spectrum in the wave number range, thus the existence of the inertia subrange was confirmed. The energy dissipation rate ε was determined from equation (9.23). It was found that the energy spectrum for 17 samples obtained from runs of 4 to 20 minutes satisfy the law of $k^{-5/3}$, when they are plotted against k on a log-log graph. Also, the values of constant C_F for a range of ε of 10^{-2} to 1 showed the average of 0.424 with a slight scattering.

In Fig. 11 the normalised spectrum log $F(k/k_s)$ averaged from 17 samples is plotted against $\log(k/k_s)$. The curve indicates that the inertial subrange exists for a range of k/k_s from 10^{-1} down to 10^{-3}.

In the presence of waves with a low background turbulence, the measurements were taken to determine the energy dissipation due to waves. The energy spectrum $E(k)$ showed some disturbances

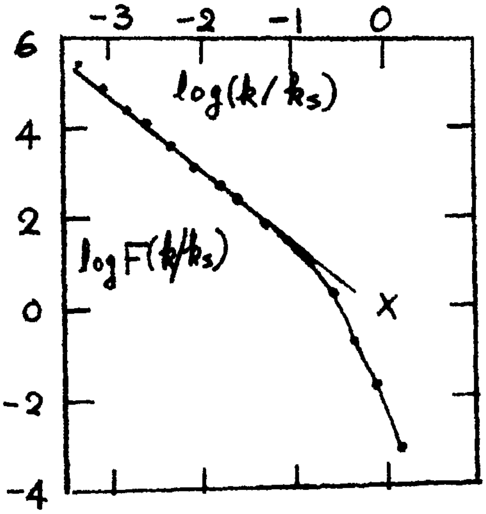


Fig. 11. The normalised spectrum function in a tidal stream. (Stewart and Grant, 1962)

due to the waves. However, the maximum dissipation ε_{max} could be determined by fitting the averaged normalized spectrum curve of Fig.11 to each of such spectrum, at a few depths down to 15 m and for wave heights of 0.1 to 0.9 meters. It is noticeable that values of ε increase with wave heights, but only slightly decrease with depth near the surface.

The energy input from the wind into the waves per unit surface was estimated from the relation:

$$E_w = \rho_a C_d W^2 \bar{c}$$

where ρ_a is the density of air, C_d is the drag coefficient, W is the wind speed and $\bar{c}$ is the mean phase speed of the waves. For $W = 7$ m/s and $\bar{c} = 3$ cm/sec, we can obtain $E_w = 100$, ergs cm^{-2} sec^{-1}, using probable values of constants. On the other hand, the energy dissipation obtained by integrating ε determined from measurements below the depth of 1 m from the surface is equal to 3 ergs $cm^{-2} sec^{-1}$, which is a full order of magnitude less than energy input estimated above. This difference leads to a conjecture that dissipation by waves is concentrated very close to the surface, essentially above the trough line. The mechanism of dissipation can be considered mainly due to breaking of waves.

The mechanism of energy dissipation by wave breaking was discussed in Chapter 6. The measurements of Grant and Stewart (1962) for the first time gave a solid basis for such analysis, though the data are still scanty and by no means complete.

10 Horizontal turbulence in large scale motion

Defant (1921) introduced the idea of horizontal mixing of momentum to dynamics of general circulation of the atmosphere. He considered that the zonal current of the atmosphere is maintained through the latitudinal momentum exchange due to travelling cyclones and anticyclones. He regarded these disturbances of synoptic scale as turbulent elements superimposed on the mean zonal currents. [illegible] comparing a solution of the stationary, linear equation of momentum transfer over the earth with the observed distribution of the zonal current of the atmosphere, he determined the eddy viscosity as $10^{\text{[illegible]}}\ \mathrm{cm^2/sec}$.

This idea was developed by several workers. Lettau (1939) applied the mixing length theory to synoptic and climatological weather charts. He determined the mixing length and horizontal eddy viscosity corresponding to the scale of general circulation.

Rossby (1941) discussed the maintenance of the atmospheric zonal flow from the idea of horizontal mixing, although he did not intend to determine the eddy viscosity. He assumed that the angular momentum is conserved by the large scale turbulent mixing in the lower latitude. Then the angular momentum becomes uniform by such mixing, and we have

$$R\cos\phi\,(u + R\Omega\cos\phi) = \Omega R^2 C_R \qquad (10\cdot1)$$

where R is the radius of the earth, ϕ is the latitude, Ω is angular velocity of the earth, u is the zonal velocity of the wind and C_R is a constant. This equation yields

$$\frac{u}{\Omega R} = \frac{C_R}{\cos\phi} - \cos\phi \qquad (10\cdot2)$$

He further considered that the absolute vorticity is conserved by such mixing in the higher latitudes. Then, the absolute vorticity becomes uniform over the high latitudes and we have:

$$-(R\cos\phi)^{-1}\frac{\partial}{\partial\phi}(u\cos\phi+R\Omega\cos^2\phi)=\text{Const.}\qquad(10\cdot3)$$

This constant can be determined by the condition at the pole, where the absolute vorticity is equal to 2Ω. Therefore, we have:

$$u(R\Omega)^{-1}=(1-\sin\phi)(1+\sin\phi)^{-1}\cos\phi\qquad(10\cdot4)$$

These two equations generally agree with the observed distribution of the zonal wind when the boundary of these two domains is taken at 45° N.

Stommel (1948) was the first to be able to explain the peculiar feature of oceanic circulation which has a strong and narrow current along the western coast. He treated analytically a model of wind-driven oceanic circulation in a rectangular ocean, considering variation of Coriolis' parameter with latitude and frictional force proportional to the mean velocity.

Munk (1950), Hidaka (1950) and Ichiye (1950d) independently introduced the eddy viscosity terms to this model as frictional forces. Munk estimated the value of eddy viscosity equal to 5×10^{7} cm^{2}/sec., by taking the width of the western current as 200 km. This width becomes equal to $\pi(A/\beta)^{1/3}$, where A is the eddy viscosity and β is the rate of variation of Coriolis parameter with latitude. Ichiye discussed the relative importance of inertia terms, β-term and horizontal mixing terms

for sinusoidal wind stresses. He concluded that inertia terms are negligible for the half wave length in the east-west direction of wind stress system longer than [illegible]00 km and the β-term is negligible for the scale (half wave length) of the wind system less than 200 km for the eddy viscosity of about 5×10^{7} cm^2/sec.

Hidaka and Iida (1957) treated the effects of Coriolis' force and of dimension of the ocean on the oceanic circulation. They found that the westward intensification becomes perceptible only when the dimensionless parameter $\beta L_S^3 / A$ exceeds 200, where L_S is the dimension of the ocean.

The large values of horizontal eddy viscosity estimated from the viscous boundary model of wind-driven oceanic circulation aroused suspicion of some workers. They contended that eddy viscosity of 10^8 cm^2/sec. corresponds to the eddies of scales of the width of the western boundary current. Charney (1955) and Morgan (1956) treated a model of inertia boundary layer for the western boundary current, completely neglecting the frictional forces and obtained the results which represent main features of the observed circulation. Ogawa (1960) discussed a model including both frictional and inertial boundary effects and applied the boundary layer techniques of the viscous flow along the solid plate. However, the problem of the relative importancy of the frictional or the inertia boundary to the western boundary to current is still to be sloved.

Rossby (1936) discussed the lateral stresses in the ocean

caused by horizontal eddies with scales between synoptic meteorological disturbances and the eddies contributing vertical mixing. Main features of his discussion consist of two parts: the effect of Coriolis' force on the lateral stresses and the role of such stresses in the dynamics of the strong ocean currents like the Gulf Stream. In the first part, he applied Prandtl's momentum transfer theory and Taylor's vorticity transfer theory (Goldstein, 1938) to deriving the lateral stress in a rotating system.

According to these theories, the lateral stress of a circular flow rotating around a vertical axis is given by:

$$\tau = -\overline{u'v'} = \rho l^2 (\partial V/\partial r \pm V)^2 \qquad (10.5)$$

in which V is tangential velocity at distance r from the axis and negative or positive sign corresponds to momentum or vorticity transfer theory, respectively. For the flow over the earth, the absolute tangential velocity V is given by:

$$V = v + \tfrac{1}{2} r f \qquad (10.6)$$

in which f is the Coriolis' parameter. Substituting (10.6) into the equations (10.5), we have

$$\tau = \rho l^2 \left(\partial v/\partial r + v/r + f\right)^2 \qquad (10.7a)$$

and

$$\tau = \rho l^2 \left(\partial v/\partial r - v/r\right)^2 \qquad (10.7b)$$

for the momentum and vorticity transfer theory, respectively. Rossby concluded that the momentum transfer theory is inadequate from two reasons. First, the lateral stress, according to this theory, does not vanish even for the straight uniform current, for which $\partial v/\partial r - v/r = 0$. Second, the local pressure gradient caused by a movement of vertical column of water

offsets Coriolis' force when the column is deep enough to make the divergency negligible and thus the horizontal turbulence caused by random movements of such columns is not affected by the rotation of the earth.

However, the question about effects of Coriolis' force on large scale horizontal turbulence is still to be solved. In fact, Ichiye (1953c, 1955b) found that irregular meanderings and eddies along the Kuroshio were often caused by passing fronts and cyclones and the movement of such disturbances is not geostrophic and is confined to the upper Ekman layer. Ichiye (1949), also treated as a stochastic process the diffusion of water masses in a field of Coriolis' force under random forces, neglecting the pressure gradients. He obtained the formulas for eddy diffusivities of x and y direction:

$$D_x = \frac{\kappa^2 \sigma_x^2 + 2 f \kappa \sigma_{xy} + f^2 \sigma_y^2}{(\kappa^2 + f^2)^2} \qquad (10\cdot 8a)$$

$$D_y = \frac{\kappa^2 \sigma_y^2 - 2 f \kappa \sigma_{xy} + f^2 \sigma_x^2}{(\kappa^2 + f^2)^2} \qquad (10\cdot 8b)$$

respectively. In these formulas, σ_x^2, σ_y^2 and σ_{xy} are the auto-correlation of x - and y - components of random forces and their cross-correlation, respectively and κ is frictional constant. These relations indicate that the eddy diffusivities are independent on the Coriolis' parameter only if the disturbing forces are isotropic $(\sigma_x^2 = \sigma_y^2)$ and there is no correlation between x - and y - components $(\sigma_{xy} = 0)$.

The other part of Rossby's theory (1936) is dynamics of the

Gulf Stream as a two-dimensional turbulent jet which was discussed by Tollmien (1926). Rossby considered that the pressure gradients in the ocean are almost balanced by the Corioli's terms and that the intense ocean current like the Gulf Stream is maintained by absorbing water-mass from the surroundings by the effect of lateral mixing. The equation of momentum transfer in x-direction (direction of the main current) becomes

$$\rho\left(\frac{\partial u^2}{\partial x}+\frac{\partial uv}{\partial y}\right)=-\frac{\partial p_n}{\partial x}+\frac{\partial \tau_{xy}}{\partial y} \qquad (10\cdot 9)$$

in which p_n is the residual pressure gradient after eliminating the pressure gradients equivalent to the geostrophic terms. The term $\partial p_n/\partial n$ is considered to be negligible. Tollmien's analysis gave that the total momentum transport by the current is constant, that the width of the current increased downstream linearly with x, distance from some initial point, and that the mass transport increases downstream as $x^{\frac{1}{2}}$.

Rossby applied the velocity distribution of such jet to the ocean with two layers, the lower of which is assumed to rest. He derived the distribution of depth H_u of the upper layer which decreases from the right side to the left side by about 1000 m, using the geostrophic relation:

$$f u = -g\,\frac{(\rho'-\rho)}{\rho'}\,\frac{\partial(H_u)}{\partial y} \qquad (10\cdot 10)$$

where ρ and ρ' are densities of the upper and lower layers, respectively. Further, he concluded that in a jet on a rotating system like the Gulf Stream, water is absorbed from the right hand side and is discharged to the left hand side, contrary to the symetrical non-rotating jet, owing to the decrease in H_u from right to left. He explained that the ejected water from the left edge of the Gulf Stream forms the counter current.

After Rossby, there were several investigations on horizontal mixing of water masses, momentum and energy across the Gulf Stream or the Kuroshio. There are some evidences that both currents, after leaving the continental shelf, absorb as well as discharge water from the left side, as the currents meander intensely (Ford, 1952; Ichiye, 1956b). Particularly, the water from 100 to 500 m deep to the left of the Kuroshio Extension is transferred to the right across the current and forms the Intermediate Water of low salinity and low oxygen in the North Pacific, (Ichiye, 1962).

Ichiye (1955b) also found an example of exchange of energy and momentum between the Kuroshio and the surroundings. Analysis of the hydrographic data of the semi-permanent cyclonic vortex found between Japan and the Kuroshio indicates that the total kinetic energy and momentum of the vortex showed an annual fluctuation similar to those of the Kuroshio upstream, with a time lag of about one month.

Later, Ichiye (1960a) applied the minimum momentum principle to the jet-type geostrophic flow in a two-layered ocean and determined the mode of the flow at the final stage which will be reached after the flow loses its energy through mixing. The width of the current becomes narrow and less than 100 km. However, he also found that the double jets are possible when energy is supplied to the main current, as occasionally observed in the Kuroshio (Ichiye, 1958b).

The viscous boundary theory of wind-driven oceanic circulation yields the eddy viscosity of $5 \times 10^{[illegible]}$ cm^{2}sec^{-1} for a representative

width of 200 km of the western boundary current. However, actual hydrographic data of the Gulf Stream indicate the width of only about 100 km which give the eddy viscosity less than $5 \times 10^6 cm^2$ [illegible]. Stommel (1955) determined the eddy viscosity using Pillsbury's (1890) data of surface currents measured by floats at 15 minutes' interval eight miles off Fowey Rocks of Florida Straits. The Gulf Stream flows almost northwards and the mean shear to the west of the maximum velocity [illegible] the station was located is 10^{-4} sec [illegible]. He computed the [illegible]ward momentum flux $\overline{uv} - \overline{u}\,\overline{v}$ where u and v are eastward and northward components of velocity. The eddy viscosity thus determined is $9 \pm 5 \times 10^5$ cm²/sec for 633 measurements and [illegible] 6×10^5 cm²/sec for 306 measurements. Therefore these [illegible] quite small compared with estimated values from the [illegible] boundary theory.

Ichiye (1951) analysed the GEK data [illegible] near Japan. Hydrographic sections observed [illegible] in March and May of [illegible] were divided into [illegible] groups bounded by 135°E and [illegible]°E and by [illegible]°E and [illegible] respectively, each group containing seven to nine [illegible]. The [illegible] GEK measurements [illegible] according to the [illegible] from the [illegible] region [illegible] correlation and mean square of [illegible] fluctuations u' and v' were [illegible]. The eddy [illegible] is determined from the relation [illegible] $\partial \overline{v}/\partial x = -\overline{u'v'}$ [illegible] directions are taken along and [illegible] to the direction of the mean flow, respectively. [illegible]

[illegible]

[illegible] 10^6 [illegible]

[illegible] 10 [illegible]

the viscous boundary theory and those determined by Stommel (1955). This difference seems to be due to the differences in the scale and period of time, over which the average processes were taken.

The energy transport from the turbulence to the mean flow $\overline{u'v'}\,(\partial\bar{u}/\partial y)$ computed from Ichiye's (1957) data are plotted in fig. 12 , which shows different features in the western and eastern regions divided by 141°E. In the western region, the energy is transferred from the mean current to the turbulence in the left side of the maximum velocity and is transferred in the reverse direction in the left side, and vice versa in the eastern region. In general, energy exchange is larger in the western region than in the eastern region. Webster (1961) also determined across the Gulf Stream off Onslow Bay, using the GEK data of 120 crossings during May and June of 1958. His result is also plotted in Fig. 12 , which shows a remarkable contrast with the result of Ichiye. Webster's data are more favorably compared with the energy transfer in the atmospheric zonal flow determined by Starr (1954). However, there is no obvious reason to consider that the energy transfer process is the same in the atmospheric zonal current and in the ocean currents. Also, disagreement between Ichiye's and Webster's result could be due to the difference in averaging process (the former using space average and the latter using time average) and in geographical locations of the areas measured, though the transfer mechanism might be quite similar in both ocean currents.

The statistical and spectral theory of turbulence described in chapter 9 was introduced to oceanography first by Stommel (1949),

who discussed the energy spectrum of turbulence in an inertial subrange (Chapter 9), using discrete wave number model. He expressed the energy spectrum and eddy viscosity in a form like:

$$\eta_n \sim \varepsilon^{\frac{2}{3}} L_n^{\frac{4}{3}} \quad , \quad v_n^2 \sim (\varepsilon L_n)^{\frac{2}{3}} \qquad (10\cdot 11)$$

respectively, where L_n and v_n are scale and velocity corresponding to the eddies of the n-th order scale.

Stommel associated the 4/3th law of eddy viscosity to an empirical formula found by Richardson's(1925) neighbor diffusivity which will be discussed later. This 4/3-th law was since then claimed to be confirmed by several authors. Inoue (1951) and Defant (1954) plotted various values of eddy viscosity and diffusivity estimated by different authors with different methods against the scales of motions corresponding to each estimation. The data cover from values assumed by Munk and Hidaka in their theory of wind-driven circulation to those determined by small scale diffusion experiments in lakes.

The application of spectral theory to a large scale oceanic circulation was made in a formal way by Miyazaki (1951) and by Ichiye (1951b) The latter considered that the energy spectrum of the oceanic turbulence must be modified from that given schematically by Stommel(1949) owing to the input of energy at certain wave numbers through the air-sea interaction and various quasi-periodic movements in the sea. Later, Ichiye (1952d) that the energy of turbulence of the global scale is supplied by the higher wave number components of wind stress over the ocean and proposed the form of eddy viscosity

$$\eta_n \sim L_n^{1/3} \left[\sum_{m=0}^{n} \tau_m L_m^{1/3} \right]^{1/3} \qquad (10\cdot 12)$$

where the τ_m is the wind stress components of wave number m. He estimated the annual variation of relative intensity of turbulence in the ocean from this formula, suggesting that the intensity has two maxima, winter and summer, in a year.

Iwata (1956) assumed that the difference of daily mean sea levels between stations at the main land of Japan and at an island 120 n. miles south of it is proportional to the geostrophic velocity of the Kuroshio flowing between two stations. He determined 12 harmonic components from two and six months series of daily mean sea level difference , respectively, and found that the square of amplitude of each harmonics is proportional to $n^{-7/3}$, where n is a frequency. He argued that this indicates the energy spectrum of large scale horizontal turbulence which is modified from ordinary spectrum of $n^{-2/3}$ owing to Coriolis' force. It is doubtful to consider the harmonic components as energy spectrum of turbulence and his argument on deriving the spectrum with Coriolis' force is not obvious. However, this approach to determining large scale oceanic turbulence by using tidal records will be very promising, if we use statistical techniques to determine the spectrum of fluctuations.

Gezentsvei (1959) discussed the dependency of the horizontal eddy viscosity on the averaging period of fluctuation velocities. He determined the eddy viscosity from the data of currents of two weeks' period at the anchored stations of 20 m and 100 m depth, using Ertel's method (1932). The averaging period was taken at 20 minutes, one hour and six hours. He found that the eddy viscosity $A = C_h T^{2/3}$, where T is the averaging time. He derived from the dimensional basis the constant C_h as, $C_h \sim \varepsilon^{1/2} \nu^{2/3}$,

where ν is the molecular viscosity. This result is different from a theory of Ogura (1952). It will be important to check the effects of averaging time on values of eddy viscosity in the ocean determined by various averaging processes.

Latun (1960) determined horizontal eddy viscosity in the Black Sea using the formula

$$A_x = \overline{(u')^2}/(\partial \bar{u}/\partial x) \; ; \; A_y = \overline{u'v'}/(\partial \bar{u}/\partial y) \qquad (10\cdot 13)$$

where x is the direction of the mean flow ($\bar{v} = 0$). He found that the values of A_x are almost constant ($\approx 3.6 \times 10^8$ cm^2/sec) and those of A_y decrease from 6.7 to 5.2×10^6 cm^2/sec for the averaging time of 5 to 60 minutes, contrary with the result of Gezentsvei (1959).

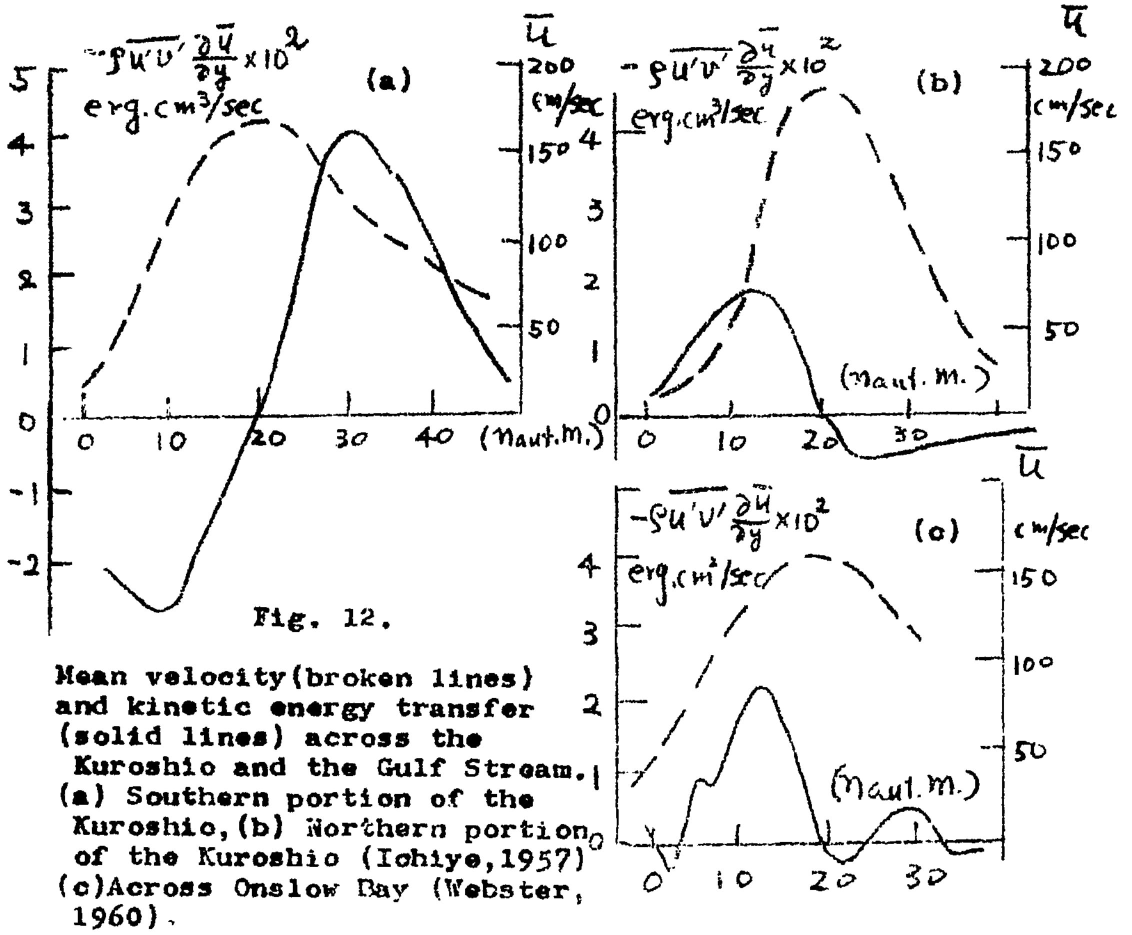


Fig. 12.

Mean velocity (broken lines) and kinetic energy transfer (solid lines) across the Kuroshio and the Gulf Stream. (a) Southern portion of the Kuroshio, (b) Northern portion of the Kuroshio (Ichiye, 1957) (c) Across Onslow Bay (Webster, 1960).

11. Turbulent diffusion. Approximate analysis of one- and two-particle problems.

Diffusion of passive quantities due to turbulence was already treated in several places, using a transport equation of Eulerian type. However, treatment of Lagrangean type is often more appropriate in some problems.

According to Batchelor and Townsend (1956), Lagrangean type treatment of turbulent diffusion can be divided into two kinds: an one-particle analysis and two-particle analysis. Taylor (1921) in his classical paper on diffusion by continuous movements laid the foundation for most of the later work on the one-particle analysis. The existing theories on turbulent diffusion are far from being complete, because of such inherent difficulties as the wide variation of statistical properties in the diffusion process and because of the analytical complications inherent to the Lagrangian description (Lin, 1958).

In the simplest form of the theory of diffusion by continuous movements, we restrict ourselves to the case in which the velocity is a stationary random function of time. We consider that all the particles are concentrated in a [illegible] plane at [illegible] = 0. If Y is the coordinate of a particle at time t, then

$$\frac{1}{2}\frac{d}{dt}\overline{Y^2} = \overline{Y\frac{dY}{dt}} = \overline{Y_t\,[illegible]} = \overline{[illegible]\int_0^{t}[illegible]} \qquad ([illegible])$$

where the average is taken in the statistical sense or over planes parallel to the x, z plane. The [illegible] stationary random function of time yields [illegible] coefficient

$$[illegible] \qquad ([illegible])$$

where $\overline{v^2}$ is independent on time. Then the relation ([illegible]) becomes

$$d\overline{Y^2}/dt = 2\overline{v^2}\int_0^t R(\xi)\,d\xi \qquad (11\cdot 3)$$

Integrating this with time, we have:

$$\overline{Y^2} = 2\overline{v^2}\int_0^t (t-\xi)R(\xi)\,d\xi \qquad (11\cdot 4)$$

In particular, for small values of ξ such that $R(\xi)$ is very close to unity,

$$\overline{Y^2} = \overline{v^2}\,t^2 \qquad (11\cdot 5)$$

When $R(\xi)$ becomes zero for sufficiently large t, we have

$$\overline{Y^2} \sim 2\overline{v^2}\,t\int_0^\infty R(\xi)\,d\xi - 2\overline{v^2}\int_0^\infty \xi R(\xi)\,d\xi \qquad (11\cdot 6)$$

This relation is valid only when the integrals do converge but there is some doubt on the validity of convergency for oceanic and atmospheric turbulence, for which stationariness in time is not obvious.

From analogy with molecular diffusion, eddy diffusivity is defined by:

$$K = \overline{v^2}\int_0^\infty R(\xi)\,d\xi \qquad (11\cdot 7)$$

Many semiempirical relations are given for various forms of the correlation functions by Frankiel (1953)

Ichiye (1960b) applied the equations ([illegible]) and (11·6) to dye diffusion experiments done by Gunnerson (1956) off the California coast. He compared the change of diameters of patches from an instantaneous source with the mean concentration derived from the above theory:

$$S = \frac{Q_0}{2\pi\overline{Y^2}}\exp\left[-\frac{(x^2+y^2)}{\overline{Y^2}}\right] \qquad (11\cdot 8)$$

and the one obtained from Joseph and Sendner's theory which will be discussed later. The results of the experiments, in which the maximum dimensions of dye patches were less than 100 m, are best fitted by the concentration function derived for the initial stage, in which $\overline{r^2} \sim t^2$. He also used the patterns of dye released from continuous sources to test the concentration function obtained by Frenkiel (1953). He found that the function for the initial stage of diffusion agrees with the observed patterns which has a longitudinal dimension less than 1 km. Gifford (in personal communication) suggested that the diffusion of patches from instantaneous sources must be explained by two-particle analysis.

The idea of two-particle analysis was for the first time introduced by Richardson (1925) to atmospheric diffusion problems. He proposed neighbor concentration in one dimension which is defined as:

$$n(l, t) = \int_{-\infty}^{\infty} s(x + l, t)\, s(x, t)\, dx \qquad (11 \cdot 9)$$

where $s(x,t)$ is the ordinary concentration. In other words $n(l)\,dl$ is the number of pairs of diffusate particles which have a distance between l and $l + dl$. Rejecting the ordinary Fickian as unsuitable, Richardson postulated the neighbor diffusion equation for n:

$$\frac{\partial n}{\partial t} = \frac{\partial}{\partial l}\left\{ F(l)\, \frac{\partial n}{\partial l} \right\} \qquad (11 \cdot 10)$$

in which is called neighbor diffusivity.

The wide use of neighbor concentration was hampered by difficulties in determining it from observations. Also, the ordinary concentration cannot be uniquely defined from n, as Ichiye (1950e) has shown. A Fourier transform of the equation becomes:

$$N(\kappa, t) = S(\kappa, t)\, S^*(\kappa, t) = |S(\kappa, t)|^2 \qquad (11 \cdot 11)$$

where $N=\int_{-\infty}^{\infty} n(\ell,t)e^{-2\pi\chi\ell i}d\ell$, $S=\int_{-\infty}^{\infty} s\,e^{-2\pi\chi x i}dx$ is the Fourier transform of n, s and S^{*} is the conjugate of S. Therefore, the neighbor concentration only defined the amplitude $|S(\chi,t)|$ but not the phase of $S(x,t)$. Shönfeld (1962) pointed out that, when the concentration $S(x,t)$ has the random variation $S'(t,x)$ the statistical average of the equation (11·11) gives:

$$\overline{N}(\chi,t)=\overline{S}(\chi,t)^2+\overline{|S'(\chi,t)|^2} \qquad (11\cdot 12)$$

Therefore, the mean neighbor concentration cannot be translated into the mean ordinary concentration if the random fluduations of the ordinary concentration are unknown.

Richardson (1926) induced the formula $F(\ell)\sim\ell^{4/3}$ from 5 sets of observations of such atmospheric diffusion as scattering of volcanic ashes or of toy balloons, for the range of length scale from 15 to 5×10^{4} m. This formula is usually referred as the 4/3 law. Later, Stommel (1949) determined the neighbor diffusivity in the ocean for the distances of 10 cm to 100 m, using the relation:

$$F\left[\tfrac{1}{2}\,\overline{(\ell_t+\ell_0)}\right]=\overline{(\ell_t-\ell_0)^2}\,/2t \qquad (11\cdot 13)$$

in which ℓ_0 and ℓ_t are the initial distances of pairs and those after time t, respectively, and bar indicates the statistical average. Olson and Ichiye (1959) determined $F(\ell)$ from the data of drift cards and drift bottles, using the formula (11·13) or modified relations. They obtained the empirical relation

$$F(\ell)=0.025\,\ell^{4/3} \qquad (11\cdot 14)$$

for the range of distances from 1 cm to 1000 km.

Batchelor (1950) applied the similarity concept to the neighbor diffusion, i.e. two particle analysis. Dimensional arguments show that the mean-square separation $\overline{\ell(t)^2}$ has a rate of change of the following form:

$$d(\overline{l^2})/dt = \varepsilon t^2 G(l_0 \varepsilon^{-\frac{1}{2}} t^{-\frac{3}{2}},\ t \varepsilon^{\frac{1}{2}} \nu^{-\frac{1}{2}}) \qquad (11\cdot 15)$$

where l_0 is the initial $(t = 0)$ separation, ν is the molecular viscosity, and G is an universal function. For turbulence with very large Reynolds number, such as in the ocean and the atmosphere, $l(t)$ and l_0 are large compared with $(\nu^3/\varepsilon)^{\frac{1}{4}}$ which is equal to the length scale of the viscous dissipation range (Chapter 9). Therefore, $d(\overline{l^2})/dt$ must be independent on ν. When t is sufficiently small, the rate of change of $\overline{l^2}$ is linear in t. Since the velocities of the two particles remain approximately constant during the diffusion, so that

$$d\,\overline{l^2}/dt \sim t\ (\varepsilon l_0)^{\frac{2}{3}} \qquad (11\cdot 16)$$

When t is large, the relative motion of the two particles no longer depends on the initial separation l_0. Then the equation (11·15) becomes:

$$d(\overline{l^2})/dt \sim \varepsilon t^2 \qquad (11\cdot 17)$$

Combining these two formulas, Batchelor obtained the relation:

$$\overline{[l^2(t) - l_0^2]}\,(\overline{l_0^2})^{-1} = k_1 (t \varepsilon^{\frac{1}{3}} l_0^{-\frac{2}{3}})^2 + k_2 (t \varepsilon^{\frac{1}{3}} l_0^{-\frac{2}{3}})^3 \qquad (11\cdot 18)$$

where k_1 and k_2 are universal constants. The equation (11·18) indicates that the relation (11·16) and (11·17) is valid for time scale $t \ll (l_0^2/\varepsilon)^{\frac{1}{3}}$ or $t \gg (l_0^2/\varepsilon)^{\frac{1}{3}}$ respectively.

The relations (11·16), (11·17) and (11·18) reveal explicitly the accelerating character of relative motion such that, as the separation increases, larger eddies may catch each particle and thus the tendency to separate becomes even more rapid.

Ichiye and Olson (1960) explained Richardson's 4/3 law on dimensional basis. First, they solved analytically the equation (11·10) by taking $F(l) = k_R l^m$. By comparing the solution of neighbor concentration with the relation (11·13), which was used

by Stommel and others to determine the neighbor diffusivity from observations, they proved that the formula (11·13) is a good approximation for $m \approx 4/3$ and for the time intervals satisfying either $|\ell^2 - \ell_0^2| \gg \ell_0^2$ (large time scale) or $|\ell^2 - \ell_0^2| \ll \ell_0^2$ (small scale). Then, using Batchelor's relation (11·18) they proved the 4/3 law. For large time scale, the equation (11·18) becomes:

$$\overline{\ell^2(t) - \ell_0^2} \sim \overline{\ell(t)^2} \sim t^3 \qquad (11\cdot19)$$

The neighbor diffusivity determined by the equation (11·13) becomes

$$F(\bar{\ell}) = (\overline{\ell^2 - \ell_0^2})/(2t) \sim t^2 \sim (\overline{\ell^2})^{\frac{2}{3}} \approx (\bar{\ell})^{\frac{4}{3}} \qquad (11\cdot20)$$

For small time scale the equation (11·13) yields the value of $F(\ell_0)$ as:

$$F(\ell_0) = (\overline{\ell^2 - \ell_0^2})/(2t) \sim t\,\ell_0^{\frac{8}{3}} \qquad (11\cdot21a)$$

However, $t \sim \ell_0^{-\frac{1}{3}}\,|\overline{\ell^2} - \ell_0^2|^{\frac{1}{2}} \approx \ell_0^{\frac{2}{3}}$, since $|\overline{\ell^2} - \ell_0^2| \sim \ell_0^2$ for small time scale. Therefore, again we have:

$$F(\ell_0) \sim \ell_0^{\frac{4}{3}} \qquad (11\cdot21b)$$

Ozmidov (1957) made extensive measurements of neighbor diffusivity, using the formula similar to (11·13), in the Caspian Sea and in an artificial reservoir of dimensional 8.6 m x 8.0 m with 3 m depth. From 507 sets and 165 sets of measurements in the sea and the reservoir, respectively, he obtained the empirical formula for neighbor diffusivity in shallow water of depth H as:

$$F(\ell, \ell/H) = C_z\,\ell^{4/3}\,f_z(\ell/H) \qquad (11\cdot22)$$

in which f_z is a correction function for the 4/3 law. The function f_z is almost equal to unity for $\ell > H$ and increases to 4 at $\ell/H \approx 0.4$. Ozmidov (1958) explained this dependence on relative depth from an hypothesis of an inclining axis of the turbulent eddies. Then he (Ozmidov, 1960) used the ordinary

eddy diffusivity of a form $\chi \sim r^{4/3}$ in a circular symmetric Fickian equation and obtained the solution of concentration such as:

$$s(r,t) \sim t^{-9/2} \exp\left(-\frac{r^{2/3}}{a_z t}\right) \tag{11·23}$$

where r is the distance from an initial point source and a_z is a constant. However, there is no obvious reason that the 4/3 law for neighbor diffusivity can be applied to the ordinary eddy viscosity.

Nan'niti and Okubo (1957) reported the result of experiment of dye diffusion and discussed the anisotropy of the patch. Ichiye (1962) did more extensive experiments on dye patches with length scale less than 100 m and concluded that the patches usually become elongated in the direction of wind or current if its speed exceeds certain critical values.

Taylor (1954) discussed the diffusion of marked flow through a pipe and derived the virtual coefficient of longitudinal diffusion arising from the combined effect of lateral diffusion and variation of velocity over the cross-section. The argument was extended to an open channel by Elder (1959).

The problem has a practical interest because one of the methods used by engineers for measuring a speed of a large channel is to release a packet of salt at one point and to measure the salinity at another point distant. The Fickian diffusion equation is given by

$$\frac{\partial}{\partial t}S + u\frac{\partial S}{\partial x} = \frac{\partial}{\partial x}\left(\chi_x \frac{\partial S}{\partial x}\right) + \frac{\partial}{\partial y}\left(\chi_y \frac{\partial S}{\partial y}\right) + \frac{\partial}{\partial z}\left(\chi_z \frac{\partial S}{\partial z}\right) \tag{11·24a}$$

in which x is the direction of mean motion and χ_i is the component of the eddy diffusivity. The velocity of mean flow u consists of

the mean velocity over the cross section, U and the deviation at individual points u'. If S is assumed to have a form $S(x-Ut, y, z)$ and longitudinal diffusion is neglected, equation (11·24a) becomes

$$u' \frac{\partial S}{\partial X} = \frac{\partial}{\partial y}\left(\chi_y \frac{\partial S}{\partial y}\right) + \frac{\partial}{\partial z}\left(\chi_z \frac{\partial S}{\partial z}\right) \qquad (11\cdot 24b)$$

in which $X = x - Ut$.

For a circular pipe, Karman's logarithmic law of velocity near the wall yields the velocity distribution

$$u = u_0 - u_* f(R) \qquad (11\cdot 25a)$$

$$f(R) = 1.35 - 2.5 \log(1-R) \qquad (11\cdot 25b)$$
$$(0.9 < R < 1)$$

where R is the ratio of distance r from the center to the radius of the pipe, a, u_0 is the velocity at the center $r=0$ and u_* is the frictional velocity $(=\sqrt{\tau_1/\rho})$ for the stress at the wall τ. Numerical coefficients in $f(R)$ are determined by use of data from experiments by Stanton and Pannell and Nikuradse. This velocity distribution yields the average velocity U and the deviation u' :

$$U = u_0 - 4.25\, u_* \; ; \; u' = u_*\left[4.25 - f(R)\right] \qquad (11\cdot 26)$$

Reynolds' analogy between transfer of matter and momentum by turbulence postulates that the eddy diffusivity is equal to the eddy viscosity. Therefore, the eddy diffusivity χ_r in r-direction is given by

$$\chi_r = \tau \Big/ \left(-\rho \frac{\partial u}{\partial r}\right) \qquad (11\cdot 27)$$

Since Karman's similarity hypothesis leads to $\tau = \tau_1 / R$, equation (11·27) becomes

$$\chi_r = a R\, u_* \left[f'(R)\right]^{-1} \qquad (11\cdot 28)$$

[illegible]

[illegible] X [illegible]

[illegible] are substituted in [illegible]

$$[illegible] \frac{d}{d\,x} \int_0^R [illegible] \,\Phi(R)\,dR$$

where $\Phi(R) = \int_0^R \{ f(R) - [illegible] \}\, R\, dR$

The rate of transfer of S due to the deviation [illegible] is given by

$$Q = 2 [illegible] \int_0^1 S_2\, u' \, R\, dR$$

If the virtual eddy diffusivity K_x in a direction [illegible] from the relation $Q = -K_x\,(\pi a^2)\,\frac{\partial S}{\partial x}$

the calculation of [illegible] from equation [illegible]

[illegible]

[illegible] diffusivity of a direction [illegible]

[illegible] the lay[illegible]

[illegible] S [illegible]

[illegible]

$$[illegible] R [illegible] \frac{\partial}{\partial x} [illegible]$$

[illegible]

at the bottom. Equations (11·30) and (11·34) with this velocity distribution and corresponding eddy diffusivity χ_z yield the virtual eddy diffusivity owing to convection and longitudinal diffusion

$$K_x = 5.86\, h\, u_* \qquad (11\cdot 32b)$$

and

$$K_x' = 0.068\, h\, u_* \qquad (11\cdot 35b)$$

respectively, in the open channel. He proved these results by experiments and obtained the virtual diffusivity

$$(K_x + K_x')_{exp} = 6.06\, h\, u_* \qquad (11\cdot 36b)$$

Ellison (1960) pointed out that Elder's velocity profile gives infinite velocity at the free surface. He proposed the shear stress τ which varies lineary with height, vanishing at the surface:

$$\tau = \rho u_*^2 = \rho u_{*0}^2 (1 - Z) \qquad (11\cdot 37)$$

From the similarity law for small turbulence near the free surface postulated that the vertical eddy viscosity near the free surface should be

$$\eta \approx m\, u_* (h - z) \qquad (11\cdot 38a)$$

where m is a constant similar to Karman's k_0. Since near the bottom of the channel

$$\eta \approx k_0\, u_{*0}\, z \qquad (11\cdot 38b)$$

the approximation to η throughout the depth of the channel, satisfying the condition (11·38a) near the surface can be given by

$$\eta (k_0 u_{*0} h)^{-1} = \zeta^3 (1 - \zeta^2)(b^2 - \zeta^2)(b^2 - 1)^{-1} \qquad (11\cdot 39a)$$

where $q = (1 - z/h)^{\frac{1}{2}}$ and $b = m^{\frac{1}{2}}(m - k_0)^{-\frac{1}{2}}$.

This formula may be compared with that resulting from Elder's logarithmic profile

$$\eta\,(k_0 U_{*0} h)^{-1} = q^2(1-q^2) \qquad (11\cdot 39b)$$

and with that resulting from an application of Karman's hypothesis:

$$\eta\,(k_0 U_{*0} h)^{-1} = 2q^2(1-q) \qquad (11\cdot 39c)$$

The velocity profiles can be easily computed from the relation $\eta(\partial u/\partial z) = \tau$. The results corresponding to $(11\cdot 39a)$, $(11\cdot 39b)$ and $(11\cdot 39c)$ are

$$\frac{k_0(u-U)}{U_{*0}} = \frac{b^2-1}{b}\ln\frac{b+1}{b-1} - \ln\frac{1+q}{1-q} + b^{-1}\ln\frac{b+q}{b-q} \qquad (11\cdot 40a)$$

$$\frac{k_0(u-U)}{U_{*0}} = 1 + \ln(1-q^2) \qquad (11\cdot 40b)$$

$$\frac{k_0(u-U)}{U_{*0}} = \frac{5}{6} + q + \ln(1-q) \qquad (11\cdot 40c)$$

The virtual longitudinal diffusivity is given by:

$$K_x = \int_0^1 [\eta(Z')]^{-1}\left\{\int_{Z'}^1 (u-U)\,dZ''\right\}^2 dZ' \qquad (11\cdot 41)$$

The comparison of Elder's empirical formula of $(K_x + K_x')_{exp.}$ with the theoretical one obtained from $(11\cdot 41)$ yields $m = 0.80$ corresponding to Karman's constant $k_0 = 0.40$

Longitudinal dispersion of discrete particles in a pipe was discussed by Batchelor and others (1955) for non-buoyant particles and by Binnie and Phillips (1958) for heavy or buoyant particles. These works gave a theoretical basis for the method of determining the discharge velocity (the average velocity over a section) by measuring the transit time of the particles between two sections

Elder (1959) applied Taylor's discussion (1954) explained above to the diffusion of discrete particles.

Taylor's one-particle diffusion theory in a circular pipe with radius a yields the equation for the Lagrangian coefficient in the longitudinal (x-direction) diffusion:

$$2 a u_* K_x t \sim \left[\overline{X(t)^2} - \overline{X(t)}^2 \right] \qquad (11\cdot42)$$

where $X(t)$ is a distance travelled by a particle during a interval t which is considered large. This equation can be converted to an expression for the dispersion of the time $T(x)$ of travel over a given distance x. For large x,

$$\overline{[X(t) - \overline{X(t)}]^2} \approx (\overline{U})^2 \; \overline{[T(x) - \overline{T(x)}]^2} \qquad (11\cdot43a)$$

and

$$t \approx \overline{T(x)} \approx \overline{U}(\alpha) \cdot x \qquad (11\cdot43b)$$

where $\overline{U}(\alpha)$ is the mean velocity determined by a transit time of a particle of radius αa. Substitution of (11·43a) and (11·43b) into (11·42) yields

$$\frac{\overline{[T(x) - \overline{T(x)}]^2}}{\left(\overline{T(x)}\right)^2} \approx \frac{2 a u_* K_x(\alpha)}{x \, \overline{U}(\alpha)} \qquad (11\cdot44)$$

Therefore, the virtual coefficient of longitudinal diffusion $K_x'(\alpha)$ can be determined by measuring the transit time $T(x)$.

Elder (1959) derived the equation for the average velocity of the particle:

$$U(\alpha) = 2 (1-\alpha)^2 \int_0^{1-\alpha} u(R) \, R \, dR \qquad (11\cdot45)$$

taking into account that the particle cannot enter the region close to the wall, $1-\alpha < R < 1$. The computation of the virtual eddy diffusivity must be modified by taking the upper limit $1-\alpha$ in equation (11·30) and using $u' = u - U(\alpha)$.

The values of $K_x(\alpha) / a u_*$ computed by Elder using Taylor's velocity distribution (1954) decrease from 10.6 for $\alpha=0$ to 1 for $\alpha=0.2$ and agree well with those determined by Batchelor and others (1955) from measurements in a pipe using equation (11·44).

The method of determining $K_x(\alpha)$ for buoyant and heavy particles is also modified by using the average velocity for such particles. The probability density $P(z)$ of the particles is determined by the diffusion equation (see Chapter 15)

$$w P(z) - \eta_p (\partial P/\partial z) = 0 \qquad (11\cdot 45)$$

where w is the settling velocity. The solution for the eddy diffusivity given by (11·39b) for an open channel is

$$P(Z) = \left(\frac{Z}{1-Z}\right)^{\beta} \frac{\sin \pi\beta}{\beta}, \qquad (Z = z/h) \quad (11\cdot 47)$$

where $\beta = w (k_0 u_*)^{-1}$. The mean velocity is given by:

$$U(\alpha, \beta) = \int_0^{1-\alpha} P(Z)\, u\, dZ, \qquad (11\cdot 48)$$

The virtual longitudinal diffusivity can be determined from equation (11·30) with $u' = u - U(\alpha, \beta)$. The result is

$$K_x(\alpha, \beta) = U^2 h^2 (7560 k_0)^{-1} (64 + 21\beta - 308\beta^2 - 210\beta^3 - 35\beta^4) \quad (11\cdot 49)$$

for $\alpha = 0$, corresponding to small particles. $K_x(0,\beta)$ rises by 0.55 % above $K_x(0,0)$ to a maximum at $\beta \approx \frac{1}{3C}$ and decreases to zero near $\beta \approx \pm 0.5$. Binnie and Phillips (1958) derived the mean velocity for a circular pipe:

$$\overline{U}(\alpha, \gamma) = (1 - B \chi_*^{-2} \gamma^2)\, U(\alpha, 0) \qquad (11\cdot 50)$$

where $\gamma = w / U(\alpha, 0)$ and B is the constant depending on α and χ_* is a dimensionless diffusivity $\chi_* = \chi / a u_*$.

This equation indicates that the $\overline{U}(\alpha, \gamma)$ is an even function of γ and thus the effect of buoyancy and settling on the average velocity is same. This was proved by their experiments in a range of $|\gamma| < 0.04$.

Batchelor and Townsend (1956) discussed the effect of molecular diffusivity on diffusion. For small time interval satisfying:

$$t - t_0 \ll \varpi \;, \quad \chi \nu^{-1} (t - t_0) \ll \varpi \qquad (11\cdot 51)$$

where χ and ν are respectively molecular diffusivity and viscosity and ϖ is the mean square of the vorticity of turbulence $\overline{\omega^2}$, turbulent and molecular diffusion are additive and the total diffusivity K is given by:

$$K^2 = \overline{Y^2} + 2\chi (t - t_0) \qquad (11\cdot 52)$$

For larger values of $t - t_0$, they derived the equation

$$\Delta = K^2 - \overline{Y^2} - 2\chi (t - t_0) \approx \frac{28}{45} \chi \varpi^2 (t - t_0)^3 \qquad (11\cdot 53)$$

Saffman (1960) proved by a rigorous treatment of diffusion equation that

$$\Delta \approx -\frac{1}{9} \chi \varpi^2 (t - t_0)^3 \qquad (11\cdot 54)$$

He also derived that, for $\varpi (t - t_0) \gg 1$, $\Delta = -a_s \chi \varpi \overline{Y^2} / \overline{v^2}$ $(11\cdot 55)$ where a_s is a constant of order unity. Since

$$\overline{Y^2} \approx \frac{3}{2} C_s \left(\overline{v^2}\right)^2 \nu^{-1} \varpi^{-2} (t - t_0) \qquad (11\cdot 56)$$

the ratio of $|\Delta|$ to the effect of molecular diffusion becomes

$$|\Delta| \left[2\chi (t - t_0)\right]^{-1} \approx \frac{2}{4\sqrt{15}} a_s C_s R_\lambda \qquad (11\cdot 57)$$

where C_s is another constant of order unity and $R_\lambda = \left(\overline{v^2}\right)^{\frac{1}{2}} \lambda \nu^{-1}$ with dissipation length $\lambda = \overline{v^2} \varpi^{-2}$. Therefore, if the Reynolds number of turbulence is sufficiently large, the effect of molecular diffusion is small compared with turbulent diffusion.

Ogura (1959) discussed the effect of a finite sampling interval on estimated properties of diffusion of particles. If $v(t)$ is a stationary random velocity, its sample mean value over the interval T is

$$\langle v(t)\rangle_T = \frac{1}{T}\int_{-T/2}^{T/2} v(t+\xi)\,d\xi \qquad (11.58)$$

The fluctuation referred to this mean velocity is

$$v'_T(t+\xi) = v(t+\xi) - \frac{1}{T}\int_{-T/2}^{T/2} v(t+s)\,ds \qquad (11.59)$$

and the auto-correlation computed from this sample is given by

$$R_T(\sigma,t) = \frac{1}{T}\int_{-T/2}^{T/2} v'_T(t+\xi)\, v'_T(t+\xi+\sigma)\,d\xi \qquad (11.60)$$

The ensemble average $\overline{R_T(\sigma)}$ of the function $R_T(\sigma,t)$ is equal to the average over t on the basis of ergodic theory. Therefore,

$$\overline{R_T(\sigma)} = \lim_{A\to\infty}\frac{1}{A}\int_{-A/2}^{A/2} R_T(\sigma,t)\,dt \qquad (11.61a)$$

Substituting (11.60) into (11.61) and mindful of the definition of the mean correlation fum tion:

$$R_\infty(\sigma) = \lim_{T\to\infty} R_T(\sigma,t) = \lim_{T\to\infty}\overline{R_T(\sigma)} = \lim_{A\to\infty}\frac{1}{A}\int_{-A/2}^{A/2} v'(t)\, v'(t+\sigma)\,dt \qquad (11.61b)$$

where

$$\bar{v} = \lim_{T\to\infty}\langle v(t)\rangle_T \quad ; \quad v'(t) = v(t) - \bar{v}$$

we have

$$\overline{R_T(\sigma)} = R_\infty(\sigma) - \frac{1}{T^2}\int_0^T (T-\xi)\left\{R_T(\xi+\sigma) + R_\infty(\xi-\sigma)\right\}d\xi \qquad (11.62)$$

The function $R_\infty(\sigma)$ can be expressed by its energy spectrum $F_\infty(n)$ by

$$R_\infty(\sigma) = \int_0^\infty F_\infty(n)\cos(n\sigma)\,dn \qquad (11.63)$$

where n is the frequency. (See Chapter). Substitution of (11.63) into (11.62) leads to

$$\overline{R_T(\sigma)} = \int_0^\infty F_\infty(n)\left[1 - \frac{\sin^2 \frac{1}{2}nT}{(\frac{1}{2}nT)^2}\right]\cos n\sigma\,dn \qquad (11.64)$$

This equation indicates that the finite observation interval sets the filter expressed by the term in brackets to the energy spectrum, cutting off to the large extent the contributions from the range of frequencies less than T^{-1}.

Following Taylor's (1921) treatment of diffusion by continuous movement, the mean square displacement of particles in a finite observation interval T is defined as

$$\overline{Y_T^2(t)} = 2\int_0^t \int_0^\xi \overline{R_T(\sigma)}\, d\sigma\, d\xi \qquad (11{\cdot}65)$$

Substitution of (11.64) into (11.65) yields

$$\overline{Y_T^2(t)} = t^2 \int_0^\infty F_\infty(n) \frac{\sin^2 \frac{1}{2}nt}{(\frac{1}{2}nt)^2}\left[1-\frac{\sin^2 \frac{1}{2}nT}{(\frac{1}{2}nT)^2}\right] dn \qquad (11{\cdot}66)$$

which indicates that the finite interval sets the same filter to the energy spectrum as in equation (11.64).

A simple Lagrangian auto-correlation function:

$$\begin{aligned} R_\infty(\sigma) &= \overline{v^2}\,(1-|\sigma|/\sigma_0) && \text{for } |\sigma| \leq \sigma_0 \\ &= 0 && \text{for } |\sigma| \geq \sigma_0 \end{aligned} \qquad (11{\cdot}67)$$

leads to the equation for $\overline{Y_T^2(t)}$:

$$\overline{Y_T^2(t)} = \Lambda^2 \phi(\tilde{t}, \tilde{T}) \qquad (11{\cdot}68)$$

where $\Lambda = \sigma_0(\overline{v^2})^{\frac{1}{2}}$ is a length scale of turbulence, $\tilde{t} = t/\sigma_0$ and $\tilde{T} = T/\sigma_0$ are respectively dimensionless diffusion time and observation interval. The function $\phi(\tilde{t}, \tilde{T})$ is the ordinary mean square of displacements averaged over all realizations and a calculation shows that $\phi(\tilde{t}, \tilde{T})$ reaches maximum as $\tilde{T} \rightarrow \infty$. The diffusion from a fixed source (like chimney) consists of two parts: fluctuations of the centers of many smoke puffs and dispersion of individual puffs. The function $\phi(\tilde{t}, \infty)$ which expresses the dimensionless width of the

smoke pattern averaged over all realizations represents the resultant effect of these two kinds of fluctuations of each particle. Therefore, it is larger than the value for a finite $\tilde{T}$.

For $\tilde{T} \leq 1$ and $\tilde{t} < 1 - \tilde{T}$, the function $\phi(\tilde{t}, \tilde{T})$ becomes

$$\phi(\tilde{t}, \tilde{T}) = \tilde{t}^2 \left[\frac{1}{3}\tilde{T} - \frac{1}{3}\tilde{t} + \frac{\tilde{t}^2}{6\tilde{T}} - \frac{\tilde{t}^3}{30\tilde{T}^2} \right] \qquad (12.69)$$

If $\tilde{T} \approx \tilde{t}$, (12.69) gives $\phi(\tilde{t}, \tilde{T}) \sim \tilde{t}^3$ for the range $\tilde{t} < \frac{1}{2}$. This value of $\phi(\tilde{t}, \tilde{T})$ is equivalent to Batchelor's (1990) formula $l(t) \sim t^3$ (11.17) for the neighbor distance $l(t)$ in the inertial subrange, because $\phi(\tilde{t}, \tilde{T})$ represents the diffusion of individual puffs under the condition $\tilde{T} \approx \tilde{t}$ and because the range $\tilde{t} < \frac{1}{2}$ corresponds to the inertial subrange.

The mean concentration of the diffusate in a finite sampling period is also modified by replacing $\overline{Y^2(\lambda)}$ in equation (12.8) by $\overline{Y_T^2(\lambda)}$ in the region of isotropic turbulence, if the average distribution of smoke particles follows a Gaussian law.

Ogura's results are important to evaluation of diffusion experiments which are usually done under the condition of a finite observation interval.

12. Turbulent diffusion as stochastic phenomena

Recent developments in theory of turbulent diffusion are divided in two general categories: one, using "stochastic model" and the other is using "hydrodynamic model". In the former category, the diffusate is considered as transported by irregular motion of particles which may or may not be a part of fluid, and the motion of the fluid itself is not rigorously specified. This category may be divided into two approaches: "heuristic" and "analytic", and both approaches are discussed in this chapter. The hydrodynamic model of diffusion is based on the Eulerian transport equation which is linear about the concentration of the diffusate. Then, by use of knowledge on statistical behavior of the motion of fluid, either some averaged conditions or spectral properties of the diffusate are derived. This model will be discussed in the next chapter.

Two heuristic approaches in stochastic model were applied to the diffusion in the ocean. Joseph and Sendner (1958) considered the probability $s(R; r, t)\, 2\pi r\, dr$ that a particle initially at the distance from an origin R is found after a time t at a circular ring with a radius from r to $r + dr$. The probability $s(R; r, t+\Delta t) 2\pi r\, dr$ that a particle is found after $t + \Delta t$ at the ring can be derived by assuming that the particle initially at R moves to a ring area $(r_1, r_1 + dr_1)$ after a time t and then moves to the ring between r and $r + dr$, after a time Δt. Assuming that those two steps are independent, we have:

$$s(R; r, t+\Delta t) = \int_0^\infty s(R; r_1, t)\, s(r_1; r, \Delta t)\, 2\pi r_1\, dr_1 \qquad (12{\cdot}1)$$

Kolmogorov's theory indicates that this functional equation

can be transformed into the Fokker-Planck differential equation:

$$\frac{\partial s(R;r,t)}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left[P(r)\, r\, \frac{\partial s(R;r,t)}{\partial r}\right] \qquad (12\cdot 2)$$

with the conditions that

$$\int_0^\infty s\,(r_1\,;\,r\,,\,t)\,2\pi r\,dr = 1 \qquad (12\cdot 3)$$

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_0^\infty (r_1 - r)\, s\,(r_1\,;r,\Delta t)\,2\pi r_1\,dr_1 = 2P(r) \qquad (12\cdot 4)$$

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_0^\infty (r_1 - r)^2\, s\,(r_1\,;\,r,\Delta t)\,2\pi r_1\,dr_1 = 2P(r)\,r \qquad (12\cdot 5)$$

The first condition represents normalisation of the probability density. The second condition (12·4) indicates that during a small time interval a particle moves with the mean velocity $P(r)$. A third condition (12·5) shows that the mean square disperson of the locations of particles increases linearly with distance from the source.

The solution of (12·2), under the condition that the total mass is concentrated at the origin at $t = 0$, is ($P_0 = P(r) = const.$)

$$s\,(r\,,t) = s\,(\,0\,;\,r,t) = M_0\,(2\pi)^{-1}\,(P_0\,t)^{-1}\,exp(-R/P_0\,t) \qquad (12\cdot 6)$$

Bourret (1959) derived the distribution function (12·3) by an intuitive method. He introduced the position entropy of the diffusate, i.e. a measure of the uncertainty as to the position of a particle whose positional probability is given by $s\,(r_1;\,r,\Delta t)$ as:

$$S\,(\Delta t) = -\int s\,(r_1\,;\,r,\Delta t)\,\ln s\,(r_1\,;\,r\,,\Delta t)\;dA \qquad (12\cdot 7)$$

in which $dA = 2\pi\, r_1\;dr_1$.

The realisation of diffusion occurs so as to make $S(\Delta t)$ minimum under the constraints represented by two conditions (12·3) and (12·4).

Using Lagrangian multipliers λ and μ , minimal condition for S yields:

$$\delta \int \{ -s \ln s - \lambda r s - \mu s \} \, dA$$
$$= \int \{ (1+\mu+\lambda r) + \ln s \} \, dA \, \delta P = 0 \qquad (12.8)$$

Since δP is arbitrary, we have:

$$s = e^{-(1+\mu)-\lambda r} \qquad (12.9)$$

Substitution of this result into the conditions (12.3) and (12.4) determines λ and μ and the equation (12.6) will be obtained.

Joseph and Sendner (1962) applied the solution to the data of experiments on diffusion of radioactive substance in a length scale from 10 to 100 km, the change of turbidity in the North Sea and the spreading of the high saline Mediterrean water in the North Atlantic. They considered that the diffusion velocity P_0 has a spectrum similar to that obtained intuitively by Stomme; (1949). Ichiye's (1960) analysis of experiments of dye patches indicates that the Joseph Sendner's solution is not appropriate for diffusion with length scale less than 100m.

Schönfeld (1962) proposed another heuristic model of horizontal diffusion. Let a particle moving with velocity v in a direction χ with x-axis transport the substance from the distance r . The concentration of the substance arriving at (x,y) is:

$$s(x - r\cos\chi, \; y - r\sin\chi)$$

The transport in the x-direction by this particle is:

$$s(x - r\cos\chi, \; y - r\sin\chi) \, v\cos\chi$$

The total transport n_x can be obtained by averaging this over all possible distances r and over all angle χ. If $\nu^* dr/r$ is considered as the statistic weight of the irregular motion of particles at the distance r, we have:

$$n_x = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\chi \int_0^\infty \frac{dr}{r}\, \phi(r)\, s(x - r\cos\chi, y - r\sin\chi)\cos\chi \qquad (12{\cdot}10)$$

where $\phi = \nu\nu^*$ is assumed to be a function of r only.

Let a Fourier transform of $s(t;x,y)$ be:

$$S(t;\lambda,\mu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\,dy\; s(t;x,y)\,e^{-2\pi i(\lambda x+\mu y)} \qquad (12{\cdot}11)$$

and N_x and N_y be Fourier transforms of n_x and n_y, respectively. If we put:

$$K(\sigma) = \frac{1}{2\pi\sigma}\int_0^\infty \frac{dr}{r}\,\phi(r)\,J_1(2\pi\sigma r) \qquad (12{\cdot}12)$$

with J_1 denoting the Bessel function, the equation (12·10) can be reduced to an involution integral:

$$N_x(t;\lambda,\mu) = -2\pi i\lambda\, K(\sigma)\, S(t;\lambda,\mu) \qquad (12{\cdot}13)$$

$K(\sigma)$ is called the "integral diffusivity" by Schönfeld.

In the absence of a mean flow, the transport equation in two dimensions takes the form:

$$\frac{\partial s}{\partial t} + \frac{\partial n_x}{\partial x} + \frac{\partial n_y}{\partial y} = q(t;x,y) \qquad (12{\cdot}14)$$

where $q(t;x,y)$ is the rate at which diffusate is supplied per unit time and per unit area. Fourier transformation of this equation yields:

$$\frac{\partial}{\partial t}S(t;\lambda,\mu) + 4\pi^2\sigma^2 K(\sigma)\, S(t;\lambda,\mu) = Q(t;\lambda,\mu) \qquad (12{\cdot}15)$$

where $Q(t;\lambda,\mu)$ is a Fourier transform of $q(t;x,y)$.

When we take into account an finite delay of the diffusate to be mixed with surroundings, the transport η_x is modified as:

$$\eta_x = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d\xi\, d\eta}{\pi^2}\;\frac{\xi}{\pi}\int_0^{\infty}\frac{d\tau}{\tau}\phi(\tau;\xi,\eta)\,s(t-\tau;x-\xi,y-\eta)$$

$$(\pi^2 = \xi^2+\eta^2) \qquad (12\cdot16)$$

Introducing an additional Fourier transform with respect to time:

$$S(\omega;\lambda,\mu) = \int_{-\infty}^{\infty} dt\; S(t;\lambda,\mu)e^{-2\pi i\omega t} \qquad (12\cdot17)$$

the equation (12·15) is transformed into:

$$2\pi i\omega S(\omega;\lambda,\mu)+4\pi^2\sigma^2 K(\omega;\lambda,\mu)S(\omega;\lambda,\mu)=Q(\omega;\lambda,\mu) \qquad (12\cdot18)$$

in which:

$$K(\omega;\lambda,\mu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\xi\, d\eta\; i\,\frac{\lambda\xi+\mu\eta}{4\pi^2\sigma^2\pi^3}e^{-2\pi i(\lambda\xi+\mu\eta)}$$

$$\cdot\int_{-\infty}^{\infty}\frac{d\tau}{\tau}e^{-2\pi i\omega\tau}\phi(\tau;\xi,\eta) \qquad (12\cdot19)$$

$$(\sigma^2 = \lambda^2+\mu^2)$$

The Fourier transformation of the ordinary advection equation yields the results similar to (12·15) or (12,8) under certain statistical conditions of velocity fields.

The form (12·19) suggests to us that the integral diffusivity $K(\omega,\sigma)$ is a measure of spectrum of $\phi=\eta\eta'$ i.e. the energy of turbulence. Schónfeld assumed that there is a correlation between ω and σ in the usual turbulent motion. Assuming that ω becomes large only around $\omega_0(\sigma)$, the energy transfer from larger to smaller eddies is given by:

$$\varepsilon \approx \sigma^{-2}\,\omega_0^3(\sigma) \qquad (12\cdot20)$$

Postulating that this energy is not constant in the ocean as in the inertial subrange, but increases with σ, he obtained $\varepsilon \sim \sigma^{\beta}$ and $\omega_0 \sim \sigma^{(2+\beta)/3}$.

By introducing the diffusion velocity:

$$w_D = \sigma^{-1}\omega_0 \sim \sigma^{-\alpha} \qquad (12{\cdot}21)$$

where $\alpha = (1-\beta)/3$, the value of $K(\omega,\sigma)$ becomes

$$K(\omega,\sigma) = \frac{w_D}{2\pi\sigma} F\left(\frac{\omega}{2\pi w_D \sigma}\right) = \frac{w_D}{2\pi\sigma} F(\omega') \qquad (12{\cdot}22)$$

from dimensional basis. Let $f(t')$ be the inverse Fourier transform of $F(\omega')$. Then $f(t')$ is considered as a retardation function, becomes $\delta(t')$ in case of no retardation, corresponding to $F(\omega') \equiv 1$.

Schönfeld considered the diffusion of an instantaneous, point source, which can be expressed by $q = \delta(x)\delta(y)\delta(t)$ and its Fourier transform is $Q(\omega;\lambda,\mu) = 1$. For this point source, he derived the distribution $s(r,t)$ for the case of no retardation $F(\omega') \equiv 1$ or of a form of $e^{-at'}H(t') = f(t')$, where $H(t')$ is Heaviside's unit function. Alternatively, he determined $f(t')$ for the normal distribution of $s(r,t)$ or for the solutions obtained from Joseph and Sendner's equation and from Fickian equation. By comparing theoretical distributions with observed data used in Joseph and Sendner's study, he concluded that the oceanic diffusion is best fitted by the curve with some retardation and with $\alpha = 0$. The value $\alpha = \frac{1}{3}$ corresponds to $\beta = 0$ which indicates the constant dissipation energy as postulated by Kolmogorov for the inertial subrange.

Goldstein (1951) generalised the theory of diffusion developed by Taylor (1921) by taking into account a tendency of random walks to persist in the same direction. Consider one-dimensional random walks which are performed by a discrete step Δx in x-direction at an finite interval Δt. Let $R(x,t)$ and $L(x,t)$ be the probability that the position x is occupied at time t by a rightward or a leftward-moving particle, respectively. If $p_+(\Delta t)$ or $p_-(\Delta t)$ is the probability that the particle moves into unchanged or reversed directions from the previous step, it is seen that:

$$R(x+\Delta x, t+\Delta t) = p_+(\Delta t)\, R(x,t) + p_-(\Delta t)\, L(x,t)$$

(12·23·a)

$$L(x-\Delta x, t+\Delta t) = p_-(\Delta t)\, R(x,t) + p_+(\Delta t)\, L(x,t)$$

(12·23·b)

Expanding the left hand sides, dividing by Δt and taking the limit $\Delta t \to 0$, we have:

$$\frac{\partial R}{\partial t} + v_p \frac{\partial R}{\partial x} = \left(\frac{v_p}{l_p}\right)(L - R) \qquad (12\cdot 24a)$$

$$\frac{\partial L}{\partial t} - v_p \frac{\partial L}{\partial x} = \left(\frac{v_p}{l_p}\right)(R - L) \qquad (12\cdot 24b)$$

where

$$v_p = \lim_{\Delta t \to 0} (\Delta x/\Delta t) \qquad (12\cdot 25a)$$

and

$$v_p / l_p = \lim_{\Delta t \to 0} \left[p_-(\Delta t)/\Delta t \right] = \lim_{\Delta t \to 0} \left[\{1 - p_+(\Delta t)\}/\Delta t \right]$$

(12·25b)

in which ℓ_p is a characteristic length of a flight of the particle. The probability $P(x,t)$ that the particle is found at a point x at time t is given by:

$$P(x,t) = R(x,t) + L(x,t) \qquad (12{\cdot}26)$$

Using (12·24) and (12·26), we have:

$$\frac{\partial^2 P}{\partial x^2} - \frac{1}{K}\frac{\partial P}{\partial t} - \frac{1}{v_p^2}\frac{\partial^2 P}{\partial t^2} = 0 \qquad (12{\cdot}27)$$

in which the diffusivity K is defined by:

$$K = \frac{1}{2} v_p \ell_p \qquad (12{\cdot}28)$$

Noble (1961) obtained the solution of the two-dimensional, circular symmetrical form of equation (12·27). By comparing the change with time of diameters of dye patches released in a lake with this solution and the solution of Joseph and Sendner's equation (12·2), he concluded that the latter solution fits the data better than the former for the diffusion with a time scale less than a few hours.

Bourret (1960) generalized the Goldstein's equation (2·27) into an integro-differential equation involving the velocity auto-correlations of the diffusate particles. Roberts (1961a) generalised Bourret's equation further.

Defining the rightward diffusing flux $J(x,t)$ by:

$$J(x,t) = v_p \left\{ R(x,t) - L(x,t) \right\} \qquad (12{\cdot}29)$$

we see that:

$$\frac{\partial P}{\partial t} = -\frac{\partial J}{\partial x}\,; \quad \frac{\partial J}{\partial t} + \frac{v_p}{\ell_p} J = -v_p^2 \frac{\partial P}{\partial x} \qquad (12{\cdot}30\,a,b)$$

On integrating equation (12·30b) over all x and integrating the resultant with t, we have:

$$\int_{-\infty}^{\infty} J(x,t)\,dx = A \exp\left(-2v_p t/\ell_p\right) \qquad (12{\cdot}31)$$

where A is constant.

If the particle is assumed to move to the right at $t=0$, then $A = v_p$. The definition of J yields the Lagrangian autocorrelation of velocity

$$M(t) = \int_{-\infty}^{\infty} v \, J(x,t)\, dx \qquad (12.32\,a)$$

which becomes:

$$M(t) = v_p^2 \exp(-2 v_p t/\ell_p) \qquad (12.32\,b)$$

The solution of equation (12.30) with the initial condition $J(x,0)=0$ can be written in the form:

$$J(x,t) = -\frac{\partial}{\partial x} \int_0^t M(t-t')\, P(x,t')\,dt' \qquad (12.33)$$

Substituting this into (12.30a) we have:

$$\frac{\partial P(x,t)}{\partial t} = \frac{\partial^2}{\partial x^2} \int_0^t M(t-t')\, P(x,t')\,dt' \qquad (12.34)$$

The three dimensional form of this equation is:

$$\frac{\partial P}{\partial t} = \frac{\partial^2}{\partial x_i \partial x_j} \int_0^t M_{ij}(t-t')\, P(x_i,t')\,dt' \qquad (12.35)$$

Bourret postulated that this equation is valid for general correlation functions:

$$M_{ij}(t') = \overline{v_i(t)\, v_j(t+t')} \qquad (12.36)$$

although he derived it only for a special form of $M(t)$ given by equation (12.32 b).

When these M_{ij} have Laplace transforms expressible as a ratio of polynomials, the proposed relation (12.35) can be expressed in differential forms. Let $\Pi(x_i, s)$ and $\Phi_{ik}(s)$ be the Laplace transforms of $P(x_i,t)$ and $M_{ik}(t)$, respectively, like

$$\Pi(x_i, s) = \int_0^{\infty} P(x_i,t)\, e^{-st}\, dt \qquad (12.37)$$

etc.

The Laplace transform of equation (12.35) becomes:

$$s\,\Pi(x_i, s) - P(x_i, 0) = \frac{\partial^2}{\partial x_i \partial x_k} \Phi_{ik}(s)\, \Pi(x_i, s) \qquad (12.38)$$

where $P(x_i, 0)$ is the initial value of $P(x_i,t)$. If

$\Phi_{ik}(s) = \psi_{ik}(s)/\varphi(s)$, where $\psi_{ik}(s)$ and $\varphi(s)$ are polynomials, the equation (12.38) can be written in the form:

$$s\,\varphi(s)\,\Pi(x_i, s) = \frac{\partial^2}{\partial x_i \partial x_j}\left[\psi_{ik}(s)\,\Pi(x_i, s)\right] + \varphi(s)\,P(x_i, 0) \qquad (12.39)$$

Applying the reverse Laplace transform to this equation, we have:

$$\varphi\left(\frac{\partial}{\partial t}\right)\frac{\partial}{\partial t}P(x_i, t) = \frac{\partial^2}{\partial x_i \partial x_j}\left[\psi_{ik}\left(\frac{\partial}{\partial t}\right)P(x_i, t)\right] + \varphi\left(\frac{\partial}{\partial t}\right)P(x_i, t)_{t=0} \qquad (12.40)$$

in which $\varphi\left(\frac{\partial}{\partial t}\right)$ and $\psi_{ik}\left(\frac{\partial}{\partial t}\right)$ mean that the argument of φ and ψ_{ik} should be replaced with the operator $\partial/\partial t$. An impulse point source at the origin can be expressed by $P(x_i, 0) = \delta(x_i)\,\delta(t)$.

Bourret gave several examples for $M_{ik}(t)$. For the Fickian diffusion, $M_{ik}(t) = D\,\delta_{ik}\,\delta_+(t)$, where δ_{ik} is the Kronecker delta and $\int_0^t \delta_+(t)\,dt = 1$. For the Goldstein equation, $M_{ik}(t) = \overline{v_k^2}\,\delta_{ik}\exp(-|t|\,v_k^2/K)$. (12.41)

For Frenkiel's correlations (Frenkiel, 1953),

$$M_{ik}(t) = \delta_{ik}\,\overline{v_k^2}\,\exp(-|t|/\lambda_k)\cos\omega_k(t) \qquad (12.42a)$$

$$\psi_{ik}(s) = \lambda_k\,\overline{v_k^2}\,\delta_{ik}\,(1 + \lambda_k s) \qquad (12.42b)$$

$$\varphi(s) = (1 + \lambda_k s)^2 + (\omega_k \lambda_k)^2 \qquad (12.42c)$$

For the turbulence described by Karman and Howarth (1938).

$$M_{11}(t) = \overline{v_{\parallel}^2}\,\exp(-\beta|t|) \qquad (12.43a)$$

$$M_{22}(t) = M_{33}(t) = \overline{v_{\perp}^2}\,e^{-\beta|t|}\left(1 - \tfrac{1}{2}\beta|t|\right) \qquad (12.43b)$$

$$M_{ij}(t) = 0 \qquad (i \neq j) \qquad (12.43c)$$

$$\psi_{11}(s) = \overline{v_{\parallel}^2}\,(s + \beta) \qquad (12.43d)$$

$$\psi_{22}(s) = \psi_{33}(s) = \overline{v_{\perp}^2}\,(s + \beta) \qquad (12.43e)$$

$$\varphi(s) = (s + \beta)^2 \qquad (12.43f)$$

Bourret also proved that, if $P(x_i, t)$ satisfied (12.35)

$$\overline{x_i x_k} = 2\int_0^t (t - t')\,M_{ik}(t')\,dt' \qquad (12.44)$$

which is a generalisation of the relation obtained by Taylor (192).

Roberts (1961a), for a one-particle analysis, generalised the equations (12·44a) and (12·44b) into the form:

$$\frac{\partial R}{\partial t} + v_p \frac{\partial R}{\partial x} = - f(J) \qquad (12\cdot45a)$$

$$\frac{\partial}{\partial t} - v_p \frac{\partial L}{\partial x} = f(J) \qquad (12\cdot45b)$$

in which $f(J)$ is a linear functional. Then (12·30b) is replaced by:

$$\frac{\partial J}{\partial t} + 2 v_p f(J) = - v_p^2 \frac{\partial P}{\partial x} \qquad (12\cdot46)$$

He postulated the functional $f(J)$ of the form

$$f(x,t) = \int_0^t dt' \int_{-\infty}^{\infty} dx' F(x-x', t-t') J(x', t') \qquad (12\cdot47)$$

Then the solution) can be written

$$\frac{\partial P}{\partial t} = \frac{\partial^2}{\partial x^2} \int_0^t dt' \int_{-\infty}^{\infty} dx' Q(x-x', t-t') P(x', t') \qquad (12\cdot48)$$

where $Q(x,t)$ is the Green's function for (This function satisfies:

$$\frac{\partial}{\partial t} Q(x,t) + 2 v_p f(Q) = \delta(x)\delta(t) \quad (\text{for } t > 0) \qquad (12\cdot49)$$

and vanishes for $t < 0$. On multiplying (12·48) by x^2 and integrating over all x, we have the equation for $\overline{x^2}$ which yields:

$$M(t) = \int_{-\infty}^{\infty} Q(x,t)\, dx \qquad (12\cdot50)$$

by comparing with (12·44). The three dimensional form of (12·48) becomes

$$\frac{\partial}{\partial t} P(\vec{x}, t) = \frac{\partial^2}{\partial x_i \partial x_j} \int_0^t dt' \int d\vec{x}' Q_{ij}(\vec{x} - \vec{x}', t-t') P(\vec{x}', t') \qquad (12\cdot51)$$

where the integration over $\vec{x}'$ is over all space. This equation is similar to equation (12·34) in predicting a Gaussian form for $P(x,t)$ with the correct variance as $t \to \infty$. However, equation (12·51) determines $\partial P/\partial t$ in terms of a weighted average of P not only over previous times as equations (12·34), but also over neighboring positions.

Roberts (1961a) also discussed the two-particle analysis. He divided the probability $B(x_1, x_2, t)\,dx_1 dx_2$ that two particles should lie in the intervals $(x_1, x_1 + dx_1)$ and $(x_2, x_2 + dx_2)$

respectively, into four parts: the probabilities that both move to the right and to the left and those that each moves in the opposite directions. Then he formed four differential equations for each probability similar to equations (13-58). Proceeding in the same way as for the one-particle analysis and taking $f(J) \sim J$ he determined $M(t)$ and derived the equation for $B(x_1, x_2, t)$.

Roberts (1961b) introduced an approximation: $\Omega_{ij}(\vec{x},t) \approx P(\vec{x},t)M_{ij}(\vec{x},t)$ according to Kraichnan's postulation (1959). For the isotropic case, he obtained the approximate solution of equation (13-61) for short times $t \ll L\, v_0$ and for large times $t \gg L\, v_0$, where L is macroscale of turbulence and v_0 is r.m.s. of turbulent velocity. His analysis indicates that the probability distribution resembles that for classical diffusion, but with a variable diffusivity which is proportional to $v_0^2 t$ for small times and which approaches $L v_0$ for large times. The probability distribution has a sharp front with the finite propagation speed unlike the classical diffusion.

Also, he gave an analytical basis for the behavior of two particles. For the separation r much smaller than L, the relative motion is governed mostly by the eddies whose length scale is of the same order of r. Therefore in a flow of the high Reynolds number, the diffusion depends on the spectrum of the turbulence in the inertial subrange, but not on that of energy-containing eddies. The neighbor diffusivity $N(r)$ introduced by Richardson (1926) is determined analytically. On assuming that $E(k) \sim v_0^2 L (kL)^{-1}$ as results from Kraichnan's approximation (1959), one finds that $N(r) \sim v_0 L (r/L)^{n}$ and that, in

consequence, $\overline{r^2} \sim L^2 (v_0 t/L)^{2/(2-n)}$. However, these are drived on the basic approximation, in which the energy — containing eddies do play a part in the relative diffusion. If this effect is excluded, the Kolmogorov's spectrum $E(k) \sim v_0^2 L (kL)^{-\frac{5}{3}}$ implies $N(r) \sim v_0 L (r/L)^{\frac{4}{3}}$ as the original postulation of Richardson (1926). Also we have $\overline{r^2} \sim L^2 (v_0 t/L)^3$. This indicates that the neighbor diffusivity $N(r)$ and the dispersion $\overline{r^2}$ are very sensitive to the spectrum of turbulence. Roberts' analysis (1961b), therefore, has presented a synthetic treatment of stochastic models and spectral (or hydrodynamic) models which will be discussed in the next chapter.

13. Hydrodynamic approach to turbulent diffusion.

Eckart (1948))discussed the process of mixing of cream and coffee, by classifying the process in three stages: first, sharp gradients are formed at the interfaces between milk and coffee, second, the area of interfaces increases and masses of cream and coffee are distorted by stirring, and finally the interfaces disappear by molecular diffusion. His idea is essentially the same as those of Obukhov, Yaglom, and others who applied Kolmogorov's inertial subrange in turbulence to diffusion of passive quantities.

Eckart started from the equation of transport of substance by action of incompressible fluid:

$$\frac{D\theta}{Dt} = \frac{\partial\theta}{\partial t} + u_i \frac{\partial}{\partial x_i}\theta = \chi \nabla^2 \theta \qquad (13\cdot 1)$$

and

$$\partial u_i / \partial x_i = 0 \qquad (13\cdot 2)$$

where χ is the molecular diffusivity, and repeated indices are interpreted as summation. Applying the operator ∇ to the equation(13·1) and multiplying $\nabla\theta$ with the resultant, we have:

$$\frac{1}{2}\frac{D}{Dt}(\nabla\theta)^2 = \chi \frac{\partial}{\partial x_j}\left(\frac{\partial\theta}{\partial x_j}\nabla^2\theta\right) - \chi(\nabla^2\theta)^2 - \frac{\partial u_i}{\partial x_j}\frac{\partial\theta}{\partial x_i}\frac{\partial\theta}{\partial x_j} \qquad (13\cdot 3)$$

Integrating this equation over a certain volume V, we have:

$$\frac{1}{2}\frac{D}{Dt}G^2 = \iint\left(\nabla\theta\cdot\frac{D\theta}{Dt}\right)_n d\sigma - \chi I^2 - S' \qquad (13\cdot 4)$$

in which the integration is taken over the surface of the volume and suffix n indicates the components of the vector $\nabla\theta$ normal to the surface. The surface integral can be neglected compared to the volume integrals: G^2 , I^2 and S

$$G^2 = \iiint (\nabla\theta)^2 \, dV \tag{13.5a}$$

$$I^2 = \iiint (\nabla^2\theta)^2 \, dV \tag{13.5b}$$

$$S = \iiint \frac{\partial u_i}{\partial x_j} \frac{\partial \theta}{\partial x_i} \frac{\partial \theta}{\partial x_j} \, dV \tag{13.5c}$$

G is the r.m.s. of gradients, I is a measure of inhomogeneity of the substance and S is a measure of activities of stirring. The equation(13.4) indicates that the molecular diffusivity always decreases the averaged gradients of concentration. However, S may be positive or negative.

Eckart discussed an example, in which $u = u(y)$, $v = w = 0$ and intial value of θ is given by $\theta = ax + by + c$. The solution of transport equation with $\chi = 0$ is given by:

$$\partial\theta/\partial y = b - at\,(du/dy)$$

which indicates that the G $(\sim |\partial\theta/\partial y|)$ increases with a time for large times. Therefore, this example shows that the S-term is negative, and that the shearing increased gradients of concentration.

Obukhov (1949) and Yaglom (1949) treated the passive quantities in the same way as structure of turbulence. Instead of G defined by(13.5a), they considered a measure of inhomogeneity in the

volume V defined by:

$$D = \frac{1}{2}\int_V \overline{(\theta')^2}\, dV \quad ; \quad \theta' = \theta - \overline{\theta} \qquad (13\cdot 6)$$

in which the bar indicates statistical average and the primed quantities are deviations from the averaged values.

Substraction of the equation for $\overline{\theta}$ from equation (13·1) yields:

$$\frac{\partial \theta'}{\partial t} + \frac{\partial}{\partial x_i}\left(u_i\,\theta' + u_i'\,\overline{\theta} - \overline{u_i'\theta'} - \chi\frac{\partial \theta'}{\partial x_i}\right) = 0 . \qquad (13\cdot 7)$$

Multiplying this equation with θ' and averaging statistically, we have

$$\frac{\partial}{\partial t}\frac{\overline{(\theta')^2}}{2} + \frac{\partial}{\partial x_i}\left(\frac{1}{2}\,\overline{u_i(\theta')^2} - \chi\,\overline{\theta'\frac{\partial}{\partial x_i}\theta'}\right) + \theta' u_i \frac{\partial}{\partial x_i}\overline{\theta} + \chi\overline{\left(\frac{\partial}{\partial x_i}\theta'\right)^2} = 0 \qquad (13\cdot 8)$$

where the condition of incompressibility is used. When this equation is integrated over the volume V, the second term (divergence) can be transferred into surface integral, which is small compared to the volume integral. Neglecting the surface integral as before, we have:

$$\frac{\partial D}{\partial t} = -\int_V \left[\,\overline{u_i'\theta'}\;\overline{\partial\theta'/\partial x_i} - \chi\overline{(\partial\theta'/\partial x_i)^2}\,\right] dV \qquad (13\cdot 9)$$

If we put $\overline{u_i'\theta'} = -K(\partial\theta'/\partial x_i)$, where K is the eddy diffusivity, we have:

$$\frac{\partial D}{\partial t} = \int_V \left[\,K\overline{(\partial\theta'/\partial x_i)^2} - \chi\overline{(\partial\theta'/\partial x_i)^2}\,\right] dV \qquad (13\cdot 10)$$

This indicates that the turbulent mixing increases and molecular diffusion decreases the inhomogeneity. The spectrum of inhomogeneities $(\theta')^2$ can be derived by heuristic treatment. Let us suppose that the deviation θ_n' is created by the n-th order velocity fluctuations (or eddies) with length scale l_n and with characteristic velocities v_n. Equation (13·9) indicates that the rate of increase of inhomogeneities due to these eddies is equal to $v_n(\theta_n')^2/l_n$, if molecular diffusion is neglected. The wave number range in which molecular diffusion

is negligible is called "convection subrange" by Batchelor (1959). When the convection subrange coincides with the inertial subrange, in which the viscous dissipation is unimportant, the inhomogeneities increased by the n-th order eddies per unit time are transferred to smaller eddies of the n+1 th order. The subdivisions of larger eddies into smaller ones are prescribed by the Kolmogoroff's theory (See Chapter 10). Finally, the inhomogeneity thus transferred to the smallest eddies (the m-th order) is dissipated by molecular diffusion. If we put $\chi = \overline{\kappa(\nabla\theta)^2}$, $\chi \sim \kappa v_m^2 l_m^{-2}$. The transfer process of the inhomogeneities is expressed by relations:

$$v_n (\theta_n')^2 l_n^{-1} \simeq v_{n+1} (\theta_{n+1}')^2 l_{n+1}^{-1} \simeq \cdots \simeq v_m (\theta_m')^2 l_m^{-1} \simeq \kappa (\theta_m')^2 l_m^{-2} \simeq \chi \qquad (13\cdot11)$$

On the other hand, Kolmogorov's theory (Chapter 10) yields in the inertial subrange the spectrum of v_n :

$$v_n \sim (\varepsilon l_n)^{1/3} \qquad (13\cdot12)$$

in which ε is the rate of dissipation of turbulence energy. Substitution of this spectrum into equation (13·11) results in:

$$(\theta_n')^2 \sim \chi \varepsilon^{-1/3} l_n^{2/3} \qquad (13\cdot13)$$

This spectrum is valid under the assumption of the coincidence of the inertial and convection subrange defined by:

$$L_0 \gg l_n \gg l_m \qquad (13\cdot14)$$

where L_0 is the length scale of energy-containing eddies. By use of the relations:

$$l_m \sim \nu / v_m \sim \kappa / v_m \; ; \; v_m \sim (\varepsilon l_m)^{1/3}$$

the length scale of the smallest eddies is given by:

$$l_m \sim (\kappa^3 / \varepsilon)^{1/4} \qquad (13\cdot15)$$

A more rigorous theory can be constructed on the basis of the above consideration. In the convection subrange, in which $l_m \ll \vec{r}_1 - \vec{r}_2 \ll L_0$, the difference $\theta(\vec{r}_1) - \theta(\vec{r}_2)$ can be considered as statistically isotropic. The structure function

$$D_\theta(|\vec{r}_1 - \vec{r}_2|) = \overline{[\theta(\vec{r}_1) - \theta(\vec{r}_2)]^2} \qquad (13\cdot16)$$

depends on $r = |\vec{r}_1 - \vec{r}_2|$, χ and ε only. Dimensional consideration leads to:

$$D_\theta(r) = a_D^2 \chi \varepsilon^{-1/3} r^{2/3}, \quad (l_m \ll r \ll L_0) \qquad (13\cdot17)$$

where a_D is a numerical constant. For $r \ll l_m$, $\theta(\vec{r}_1 + \vec{r}) - \theta(\vec{r}_1) \sim \vec{r}\,\nabla\theta$. Therefore, $D_\theta(r) \sim r^2$. More detailed consideration lead to the formula

$$D_\theta(r) = \frac{1}{3} \chi \kappa^{-1} r^2, \quad (r \ll l_m) \qquad (13\cdot18)$$

The value of l_m can be defined as the point of intersection of the two formulas (13·17) and (13·18), i.e.

$$a_D^2 \chi \varepsilon^{-1/3} l_m^{2/3} = \frac{1}{3} \chi \kappa^{-1} l_m^2 \qquad (13\cdot19a)$$

or

$$l_m = (27 a_D^6 \kappa^3 / \varepsilon)^{1/4} \qquad (13\cdot19b)$$

Then, the function $D_\theta(r)$ can be expressed in the form,

$$D_\theta(r) = C_\theta^2 r^{2/3} \quad (\text{for } r \gg l_m) = C_\theta^2 l_m^{2/3} (r/l_m)^2 \quad (\text{for } r \ll l_m) \qquad (13\cdot20)$$

where the three dimensional spectral density of the structure function $D_\theta(\vec{r})$ is defined as:

$$D_\theta(\vec{r}) = 2 \iiint_{-\infty}^{\infty} [1 - \cos(\vec{k} \cdot \vec{r})] \Phi_\theta(\vec{k}) d\vec{k} \qquad (13\cdot21a)$$

The isotropic condition for $D_\theta(\vec{r})$ leads to:

$$D_\theta(r) = 8\pi \int_0^\infty [1 - \sin kr/(kr)] \Phi_\theta(k) k^2 dk \qquad (13\cdot21b)$$

The function $\Phi_\theta(k)$ can be obtain od by a general theory of Fourier integral. For $D_\theta(r) \sim r^{2/3}$, we have $\Phi_\theta(k) \sim k^{-11/3}$ (13·22a)

The one-dimensional spectral density $V_\theta(k)$ is given by:

$$\Phi_\theta(k) = -(2\pi k)^{-1} (d V_\theta / dk) \qquad (13\cdot23)$$

Therefore, we have:

$$V_\theta(k) \sim k^{-5/3} \qquad (13\cdot22b)$$

Two formulas (13·23) and (13·22b) express the three- and one-dimensional spectral density of $\overline{(\theta')^2}$ in the convection subrange, respectively.

In the range of wave number larger than l_m^{-1}, $\Phi_\theta(k)$ is expected to decrease more rapidly than $k^{-11/3}$, because, for $r \ll l_m$, $D_\theta(r) \sim r^2$ and thus slope of $D_\theta(r)$ with r-axis is more steep in this range than in the convection range. The simple spectral density:

$$\begin{aligned} \Phi_\theta(k) &\approx k^{-11/3} \quad (\text{for } k < k_m) \\ &= 0 \quad (\text{for } k > k_m) \end{aligned} \qquad (13\cdot24)$$

is found to yield the structure function which is proportional to r^2 for $r \ll k_m^{-1}$ by substituting into the integral of equation (13·21b). Comparison of this integral with the expression (13·20) for $D_\theta(r)$ gives the relation between k_m and l_m:

$$k_m\, l_m \approx 5\cdot5 \qquad (13\cdot25)$$

The upper limit L_0 of the convection range can be determined from the average distribution. In the range $r \ll L_0$, the structure function $D_\theta(r)$ is mostly contributed by eddies of the size r, as explained above. When r exceeds L_0, contributions from the difference of $\overline{\theta}$ at the distance r $(\sim |\nabla\overline{\theta}|^2 r^2)$ becomes predominant over those from eddies. Therefore, the upper limit L_0 can be defined as the distance at which these two contributions are equal. Thus

$$C_\theta^2 L_0^{2/3} = |\nabla\overline{\theta}|^2 L_0^2 \qquad (13\cdot26)$$

Then we have:

$$L_0 = \left[C_\Theta^2 \, (\nabla \bar{\theta})^{-2} \right]^{\frac{3}{4}}$$

$$= \left[a_D^2 \, \chi \, \varepsilon^{-1/3} (\nabla \bar{\theta})^{-2} \right]^{\frac{3}{4}} \qquad (13\cdot 27)$$

which can be determined by measuring the average distribution of Θ .

The above discussion is rigorously valid only when the conventional subrange $(L_0^{-1} \ll k \ll l_m^{-1} = \varepsilon^{\frac{1}{4}} \chi^{-\frac{3}{4}})$ coincides with the inertial subrange $(L_0^{-1} \ll k \ll \varepsilon^{\frac{1}{4}} \nu^{-\frac{3}{4}})$. This is the case when $\nu \approx \chi$. Then one-dimensional spectrum $\Gamma(k)$ of $\overline{(\Theta')^2}$ (Inhomogeneity) and $E(k)$ of $\overline{(U')^2}$ (turbulent velocity) are given by:

$$\Gamma(k) \sim \chi \, \varepsilon^{-\frac{1}{3}} k^{-\frac{5}{3}} \qquad (13\cdot 28a)$$

and

$$E(k) \sim \varepsilon^{\frac{2}{3}} k^{-\frac{5}{3}} \qquad (13\cdot 28b)$$

respectively.

When $\nu \gg \chi$, the convection subrange is more extensive than the inertial subrange. In the part of the convection subrange beyond the inertial subrange specified by:

$$(\varepsilon \nu^{-3})^{\frac{1}{4}} \ll k \quad (\varepsilon \nu^{-1} \chi^{-2})^{\frac{1}{4}} \qquad (13\cdot 29)$$

the effect of visocisty becomes important and neither the velocity spectrum (13.28b) nor the $\overline{(\Theta')^2}$ spectrum is valid. The analysis of Batchelor (1959) shows that:

$$\Gamma(k) = -\frac{\chi}{\gamma k} \exp\left(\frac{\chi k^2}{\gamma}\right) \qquad (13\cdot 30)$$

where $\gamma \approx -0.5 \, (\varepsilon \nu^{-1})^{\frac{1}{2}}$ in which γ is an average value of the least principal rate of strain. Since $| k^2 \chi \gamma^{-1} | \ll 1$ we have:

$$\Gamma(k) \sim \chi \, (\gamma k)^{-1} \qquad (13\cdot 31)$$

The schematic representation of spectra of $\overline{(\theta')^2}$ and $\overline{(u')^2}$ is given in fig. 13 a.

When $\nu \ll \chi$, the $\overline{(\theta')^2}$ spectrum begins to fall off more rapidly than $k^{-5/3}$ for wave number larger than $(\varepsilon \chi^{-3})^{1/4}$ owing to the effect o f the moelcular diffusion. The analysis of Batchelor and others (1959) shows that $\Gamma(k) \sim k^{-17/3}$ (13-32) On the other hand, for the velocity spectrum the relation $E(k) \sim k^{-\frac{5}{3}}$ is valid in the entire inertial subrange as indicated in Fig. 13 b.

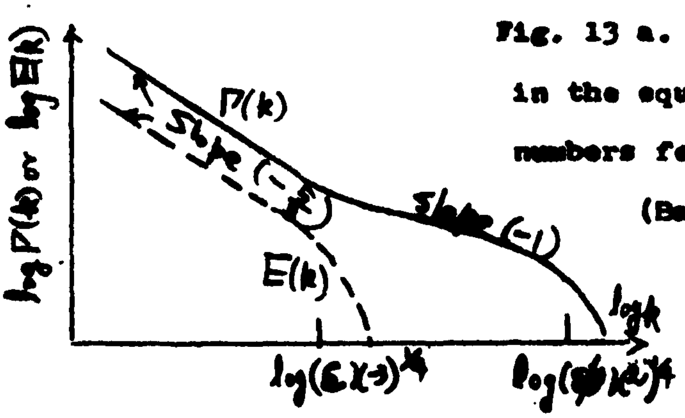


Fig. 13 a. Spectra of $\overline{(\theta')^2}$ and $\overline{(u')^2}$ in the equilibrium range of wave numbers for the case $\nu \gg \chi$.
(Batchelor, 1959)

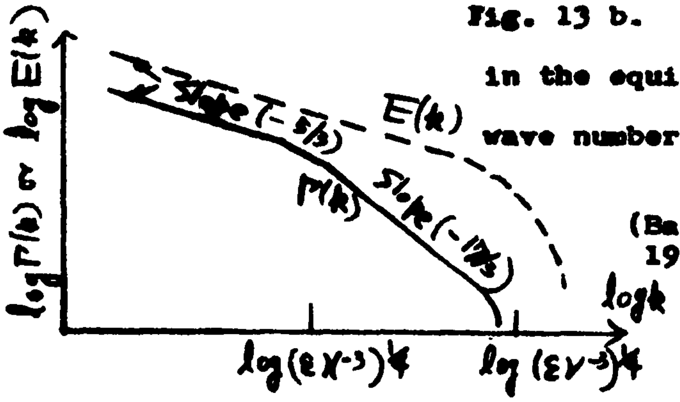


Fig. 13 b. Spectra of $\overline{\theta'^2}$ and $\overline{u'^2}$ in the equilibrium range of wave numbers for the case $\nu \ll \chi$.
(Batchelor and others, 1959)

14. Diffusion theory of tidal flushing.

The subject of disposal of a pollutant in water is an important application of the diffusion theory and here a problem of tidal flushing is discussed. There are several methods in determining the rate of flushing at various types of estuaries and harbors (Cline and Fisher, 1959).

Classical tidal prism method is suitable to a small estuary in which the tides have the uniform phase. The tidal prism is defined as the difference between the volumes of water at mean hight ide and mean low tide. The basic assumptions of the theory are: (1) during each tidal cycle the tidal prism is replaced by a new supply of water which mixes completely and uniformly with the water present in the estuary at low tides, (2) the volume of water moved seaward during the ebb tide is not returned on the following flood tide.

Let P and V be the tidal prism volume and low tide volume of the estuary. If σ is the salinity of the ocean and R is a volume of fresh water discharged during one tidal cycle, the total volume $P+R$ contains the salinity σP. Therefore, the salinity of the estuary is $P\sigma/(P+R)$ in the steady state.

Ketchem (1951) and Stommel (1952) suggested a modification of the tidal prism method by dividing an estuary into a number of segments, on the ground that the original method over estimates the flushing rate because of the assumption that mixing is complete over the entire estuary during each tidal cycle. Since Stommel's treatment is more general in nature, his

discussion is reviewed here.

The estuary is divided into a number of segments, in each of which mixing is complete at the high tide. The low tide volume and tidal prism of each segment are denoted by V_n and P_n respectively. The innermost or zero segment is defined as the one where $P_0 = R$. Let f_n and g_n be the percentage concentration of the fresh water (freshness) in the n th segment at the high and low tide, respectively. From the definition of the zero segment, $f_0 = g_0 = 1$.

Let U_n be the seaward volume flux across a landward boundary into the n th segment during the ebb tide. Then

$$U_n = \sum_{i=0}^{i=n-1} P_i \qquad (14\cdot 1)$$

Then the seaward flux during ebb or flood tide of volume and fresh water flux across landward and seaward boundaries of the n th segment is as follows:

	Landward boundary		Seaward boundary	
	Volume Flux	Fresh Flux	Volume Flux	Fresh Flux
Ebb:	U_n	$U_n f'_{n-1}$	U_{n+1}	$U_{n+1} f'_n$
Flood:	$-(U_n - R)$	$-(U_n - R) g'_n$	$-(U_{n+1} - R)$	$-(U_{n+1} - R) g'_n$

where the freshness of the water flowing in the flood tide and in the ebb tide is g'_n and f'_n, respectively. Because of the assumption that mixing is complete at high tide, $f_n = f'_n$.

The flux of fresh water volume which passes across the landward and seaward boundaries during each tidal cycle is equal to the river discharge R in the same period. This condition of conservation of fresh water volume yields

$$U_n f_{n-1} - (U_n - R) g'_n = R \qquad (14\cdot 2a)$$

and

$$\overline{U}_{n+1} f_n - (\overline{U}_{n+1} - R) g'_{n+1} = R \qquad (14 \cdot 2b)$$

at the landward and seaward boundary, respectively.

The total fresh water volume at high and low tide is $(P_n+\overline{V}_n)f_n$ and $V_n g_n$, respectively. By conservation of fresh water, the change in fresh water volume during the ebb is:

$$g_n \overline{V}_n - f_n (P_n + \overline{V}_n) = \overline{U}_n f'_{n-1} - U_{n+1} f'_n \qquad (14 \cdot 3a)$$

and during the flood is:

$$f_n (P_n + \overline{V}_n) - g_n \overline{V}_n = -(\overline{U}_n - R) g'_n + (\overline{U}_n - R) g'_{n+1} \qquad (14 \cdot 3b)$$

where g_n and g'_n are in general different and equal only when mixing is complete on low tide.

A number of different results may be obtained depending upon the nature of the mixing process involved. Model 1: No mixing occurs at low tide so that the water flowing across each boundary on the flood is unchanged. Then $g'_{n+1} = f_n$. Substitution of this into equation(142a) yields $f_n = 1$. Model 2: Mixing is complete at low tide and the segments are so large that in the limit $g'_n = f_n$. If the segment m is in the ocean, $g'_m = f_m = 0$. Thus, from equation (14·2a)

$$\overline{U}_m f_{m-1} = R, \qquad \overline{U}_m f_{m-2} = (\overline{U}_{m-1} - R) f_{m-1} + R \quad \text{etc.} \qquad (14 \cdot 4)$$

Model 3: If all the water that moves on the flood is ocean water in the form of a wedge, then $g'_n = 0$ and $\overline{U}_{n+1} f_n = R$

Model 4: Mixing is complete at low tide as well as at high tide, thus $g'_n = g_n$

Eliminating g_n and g'_n from(142)and(143), we have a recurrence formula:

$$\overline{U}_n (\overline{V}_n - \overline{U}_n + R) f_{n-1} - (\overline{U}_n - R)(P_n + V_n - \overline{U}_{n+1}) f_n = R \overline{U}_n \qquad (14 \cdot 5)$$

Ketchum's method is essentially to define each segment by

imposing to the low tide volume a condition like:

$$V_n = \sum_{0}^{n-1} P_i + V_0 = U_n + V_0 \qquad (14.6)$$

He also specified the mixing process by postulating that the total fresh water volume at high tide is given by:

$$f_n \; (P_n + V_n) = P / R_n \qquad (14.7)$$

where R_n is the exchange ratio defined by:

$$R_n = P_n \; / \; (P_n + V_n) \qquad (14.8)$$

These different models can be illustrated by a simple hypothetical example, in which four segments are determined by use of Ketchum's method. For each segment, the values of f_n are determined from Ketchum's method, and from Model 3 and 4. Model 1 yields $f_n = 1$, and Model 2 cannot be applied to the present example. The results are tabulated in the following table.

n	0	1	2	3	4
Pn	1	2	2	3	Ocean
Vn	2	3	5	7	Ocean
Un	0	1	3	5	8
fn (Ketchum)	1	1/2	1/2	1/3	0
fn (Model 3)	1	1/3	1/5	1/8	0
fn (Model 4)	1	107/135	8/15	1/8	0

This table shows that different hypotheses regarding the mixing process result in quite different salinity distribution. This is partly due to the condition that the division of the segments by Ketchum's definition is not rigorously applicable to model 1 to 4. Ambiguity on the mixing process is inheritant to such discrete models and we can avoid such ambiguity through treatment of diffusion as a process in continuous media.

Arons and Stommel (1951) discussed the steady state distribution of salinity along the longitudinal axis of an estuary. They considered that the horizontal mixing is produced by tidal currents which exercise as turbulence elements with tidal excursion (average distance excursed by water particle during one tidal cycle) as the mixing length.

Consider an estuary of uniform width b, depth H and length L. The origin of x-axis is taken at the river inflow. At the seaward end $x = L$, the salinity S is maintained at that of the ocean σ. The mean velocity U of water in the estuary is assumed to be due to the river discharge R and thus $U = R\,(bH)^{-1}$.

In an estuary whose length is small compared to a quarter tidal length, the tide is uniform over the entire estuary and its height is given by $\zeta = \zeta_0 \cos \omega t$. The tidal current V determined from the equation of continuity is given by:

$$V = V_0 \sin \omega t \; ; \; V_0 = \zeta_0 \, \omega (x/H) \qquad (14 \cdot 9)$$

Horizontal displacement of particles is obtained by integrating V with time:

$$\xi = \xi_0 \cos \omega t \; ; \; \xi_0 = -\zeta_0 x/H \qquad (14 \cdot 10, a, b)$$

In analogy with the mixing length theory, the horizontal eddy diffusivity A is considered as proportional to $V_0 \xi_0$, thus:

$$A = 2 B \zeta_0^2 \omega x^2 H^{-2} \qquad (14 \cdot 11)$$

where B ic a non-dimensional constant.

The diffusion equation for salinity S in a steady state becomes:

$$U(\partial S/\partial x) = \partial (A\, \partial S/\partial x)/\partial x \qquad (14 \cdot 12)$$

The boundary conditions are that

$$S = 0\ ,\ A\,\partial S/\partial x = 0 \quad (\text{at } x = 0)$$

$$S = \sigma\ , \qquad (\text{at } x = L) \qquad (14{,}13\,a,b)$$

By introducing non-dimensional distance $\lambda\ (= x/L)$ and flushing number ,

$$F = \bar{U} H^2 \,(2B\,\xi_0^2 L\,\omega)^{-1} \qquad (14\cdot14)$$

the solution of (14·12) with the boundary conditions (14·13) is expressed by:

$$S/\sigma = \exp\{F(1 - 1/\lambda)\} \qquad (14\cdot15)$$

Although observed salinity data for Alberni Inlet, Vancouver Island and the Raritan River, New Jersey can be fitted by theoretical curves of (14·15) using suitable flushing numbers, the numerical values of B differ by an order of magnitude for the two cases.

Stommel (1952) determined A for the Severn Estuary and the Raritan River using equation (14·12) with observed data of S and $\bar{U}$. He found that the mixing length estimated from the values of A is more nearly the order of the depth than that of the tidal excursion. This is reasonable, since the actual mixing during one tidal cycle occurs in the form of vertical mixing of the upper fresh water and lower saline water.

Maximon and Morgan (1953, 1955) discussed a model which might be called "semi-continuous" model. The estuary is assumed to consist of upper and lower layer. In their simple model, the lower layer is at rest and each layer has vertically uniform salinity. There is no horizontal mixing. Mixing of the two layers occurs only at low tide and high tide. Therefore, during flood tide and ebb tide the upper water moves without

change in its salinity (See Fig. 14.)

Let h_1, h_2 and $S_H(x)$, $S_L(x)$ be the depth of lower and upper layer and salinity at a section x after mixing at high tide and low tide, respectively. The average tidal excursion ξ is approximately equal to $2 x \zeta / h_2$, where ζ is the tidal amplitude. At low tide just before mixing, the water in the upper layer at a section x came from $x-\xi$ and thus has a salinity $S_H(x-\xi)$, while salinity of the water in the lower layer at x is $S_H(x)$. Therefore, the continuity of salt at low tide mixing yields

$$h_1 S_H(x) + h_2 S_H(x-\xi) = (h_1+h_2) S_L(x) \qquad (14\cdot16a)$$

and at high tide mixing

$$h_1 S_L(x) + h_2 S_L(x+\xi) = (h_1+h_2) S_H(x) \qquad (14\cdot16b)$$

The landward salt transport during the flood tide and the seaward salt transport during the ebb tide are approximately equal to

$$h_1 \int_0^{\xi} S_L(x+l)\,dl \sim h_1 \xi S_L(x+\tfrac{1}{2}\xi) \qquad (14\cdot17a)$$

and

$$h_1 \int_0^{\xi} S_H(x-l)\,dl \sim h_1 \xi S_H(x-\tfrac{1}{2}\xi) \qquad (14\cdot17b)$$

respectively. Therefore, the net landward salt transport during a tidal cycle across a section at x is

$$Q = h_2 \xi \left[S_L(x+\tfrac{\xi}{2}) - S_H(x-\tfrac{\xi}{2}) \right] \qquad (14\cdot18)$$

Equations (14·16a) and (14·16b) can be written

$$h_1 S_H(x+\tfrac{1}{2}\xi) + h_2 S_H(x-\tfrac{1}{2}\xi) = (h_1+h_2) S_L(x+\tfrac{1}{2}\xi) \qquad (14\cdot19a)$$

$$h_1 S_L(x-\tfrac{1}{2}\xi) + h_2 S_L(x+\tfrac{1}{2}\xi) = (h_1+h_2) S_H(x+\tfrac{1}{2}\xi) \qquad (14\cdot19b)$$

respectively. Substituting (14.19a) into (14.18) we have:

$$Q = \frac{h_1 h_2 \xi}{h_1 + h_2}\left[S_H\left(x + \tfrac{1}{2}\xi\right) - S_H\left(x - \tfrac{1}{2}\xi\right)\right]$$

$$= \frac{h_1 h_2 \xi^2}{h_1 + h_2}\frac{\partial}{\partial x} S_H(x) \qquad (14.20a)$$

Similarly substitution of (14.19b) into (14.18) yields:

$$Q = \frac{h_1 h_2 \xi^2}{h_1 + h_2}\frac{\partial}{\partial x} S_L(x) \qquad (14.20b)$$

The average salinity during a tidal cycle is defined by

$$\bar{S}(x) = \frac{1}{2}\left[S_H(x) + S_L(x)\right] \qquad (14.21)$$

Then from ( ) and ( ), we have:

$$Q = \frac{h_1 h_2}{h_1 + h_2}\frac{\partial}{\partial x}\bar{S}(x) \qquad (14.22)$$

Since it is assumed that there is no net increase of salt over a tidal cycle, this net upstream salt flux must equal the net downstream salt flux due to the river discharge R, so that

$$R\,\bar{S}(x) = \frac{h_1 h_2 \xi^2}{h_1 + h_2}\frac{d}{dx}\bar{S}(x) \qquad (14.23)$$

This equation becomes the same as Arons and Stommel's equation (14.12), if we take:

$$B = h_1 (h_1 + h_2)^{-1}$$

This derivation is interesting in a sense that the basic assumption of occurrence of vertical mixing only leads to the net horizontal transport similar to the one due to eddy diffusivity. Maximon and Morgan (1953, 1955) also treated general cases with more rigorous mathematics, but the results obtained are essentially the same as discussed above.

Dorrestein (1960) treated the non-steady diffusion process in an estuary. He divided the estuary into a number of segments

to each of which diffusion equation in different forms is applied. The change of concentration during one tidal cycle expressed by a matrix whose components are discrete Green's functions for the diffusion equation. He applied this method to the European Wadden Sea.

Ichiye (1952j) discussed a stationary two-dimensional distribution of salinity in a rectangular estuary, by solving the diffusion equation with advection terms due to irrotational velocities in the form of double Fourier series. Miyazaki (1952) discussed a solution in infinite series of non-steady horizontal diffusion equation with advective currents changing periodically with time. Ichiye (1952e) solved the non-steady diffusion equation with vertical mixing term and periodically changing advection term in a finite form. He compared the solutions with the data of fluctuations of temperature and salinity due to tidal currents.

Ichiye (1950f) also treated the non-steady diffusion process along the long axis of an estuary by solving the equation

$$\frac{\partial S}{\partial t} = \partial (A\, \partial S/\partial s)/\partial x - q(x,t) \qquad (14{\cdot}24)$$

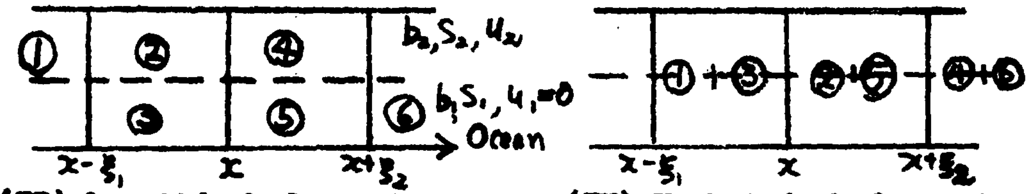
in which $A \sim U_0 \xi_0 \sim x$, considering the condition of an estuary where average longitudinal components of tidal currents is proportional to $x^{\frac{1}{2}}$ owing to the variable cross sections. He compared the solution of equation (14·24) with the annual variation of water temperatures in an estuary of Japanese coast, and concluded that estimate values of A are comparable with those determined from observed data of U_0 and ξ_0 .

Ichiye and others (1961) derived a solution of the longitudinal diffusion equation with an advection term consisting of stationary and periodic currents and applied it to qualitative explanation of salinity change in two estuaries on the northern Gulf coast. Hayami discussed the tidal mixing in a channel connecting two seas(1957). He assumed that the two water masses of each sea with salinity S_1 and S_2 start to mix at x=0 at the time t=0, say, at the beginning of the flood. Then the salinity distribution along the x-direction, long axis of the channel, will be given by a solution of the one dimensional diffusion equation:

$$S = \frac{1}{2}(S_1+S_2) + \frac{1}{2}(S_1-S_2)\,\mathrm{erf}\left[\frac{x-\xi}{2(At)^{\frac{1}{2}}}\right] \qquad (14.25)$$

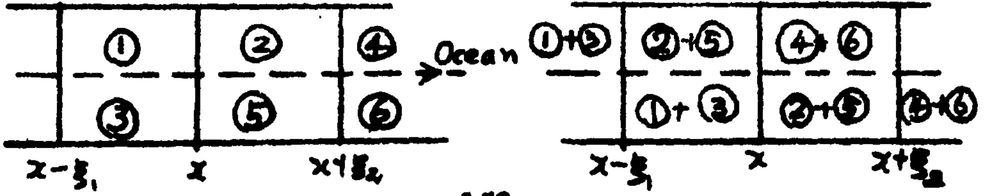
where ξ is the coordinate of the initial boundary of the two water masses and is expressed by $\xi = \int_0^t u\,dt$, and A is the horizontal diffusivity. He determined A by fitting the curve (14.25) to the observed average salinity along channels and obtained the relation $A = 0.13\bar{u}\,\xi_0$, where $\bar{u}$ and ξ_0 are the average velocity of tidal currents and tidal excusion, respectively. He seemed to assume that the mixing is completed during a half tidal cycle, although his analysis did not show it explicitly.

Fig. 14. Schematic diagram showing motion of water due to tides and mixing. (Maximon and Morgan, 1953)



(I) High tide after mixing (III) Low tide after mixing

(II) Low tide before mixing (IV) High tide before mixing



15. Turbulence and sediment transportation.

It was already recognised that sediment movement is intimately associated with turbulence. Although in earlier days many attempts were made to develop simple formulas of sediment transportation from needs of engineering problems, such objective met with little success, because basic phenomena involved, particularly by turbulence, were not fully understood yet.

Recent programs in studies of sediment transport mechanisms were summarised by Kalinske (1943) and Vanoni (1953). Careful observations of sediment transport by moving fluid indicate that there are three distinct processes: surface creep, saltation and suspension. When the vlocity of the fluid exceeds a certain limit, individual grains begin to roll or slide intermittently in isolated spots of the bed. As the vlocity increases, some grains are slightly lifted off the bed and execute saltations as they are carried downstream. As the vlocity increases further, the saltations increase in length and height and some of the grains are caught by eddies and are suspended in the current. Further increase of velocity causes much sediments in suspension and also the material moves in random patterns of billowing streaks of clouds.

The condition at which sediments just start to move is not only interesting from hydrodynamics, but also of practical importance. White (1940) derived the critical fluid shear stress τ_c for starting sediment movement by considering the balance between the drag of the fluid on particles and the gravity force. The drag is given by $\tau_0 d^2/\beta$, in

which τ_0 is the fluid shear stress at the bed, d is the diameter of the grain and β is a packing coefficient such that d^2/β is the average bed area occupied by each grain. The gravity force is $g(\rho_s - \rho)\frac{\pi}{6}d^3$ where ρ_s and ρ are the densities of the sediment and fluid respectively. Taking moments about the point of support, we get:

$$\tau_c = \beta \frac{\pi}{6} (\rho_s - \rho) g d \tan\theta \qquad (15\cdot1)$$

when θ is the angle of repose of grains. White analyzed data of his experiments and found that τ_c substancially decreases when the flow around the grains is turbulent. He used d/δ as a parameter representing the intensity of turbulence, where $\delta = 11.6\, \nu / u_*$ and $u_* = (\tau_0/\rho)^{\frac{1}{2}}$. Shields (1936) expressed the results of his experiments in a form:

$$\tau_c / [(\rho_s - \rho) g d] = f\left(\frac{u_* d}{\nu}\right) \qquad (15\cdot2)$$

in which f is a universal function and $u_* d/\nu = R_d$ is Reynolds number. He found that the function has a minimum of 0.035 in the neighborhood of $R_d = 10$ and it approaches asymptotically to 0.06 for larger values of R_d Kurihara (1948) applied the statistical theory of turbulence to explaining White and Shields' semi-empirical formulas.

The bed-load transport begins when the shearing stress at the bed τ_0 exceeds its critical value τ_c and the rate of transport Q_s increases with the stress τ_0. Therefore, DuBoys (1879) already derived the expression for Q_s in terms of τ_0 and τ_c as:

$$Q_s = B\, \tau_0 (\tau_0 - \tau_c) \qquad (15\cdot3)$$

in which β is a coefficient. Many experiments have been made to determine β and τ_c and various formulas thus obtained indicate only our ignorance on basic mechanism of bed-load transportation.

Einstein (1942) introduced the idea that the grains move in steps or jumps when the local lift exceeds the gravity. The probability that a grain on the bed begins to move during a unit time interval is expressed in two ways; as a function of transport rate Q_s, dimension and weight of the particle and settling velocity w_s as a function of the ratio of gravity force on the grain to the shearing stress of the fluid τ_0. Therefore, two non-dimensional parameters derived from quantities considered in each expression of the probability might have a functional relation. These parameters φ and ψ can be expressed by:

$$\varphi = Q_s (\rho_s/\rho) \left(g \frac{\rho_s-\rho}{\rho}\right)^{-1} d^{-\frac{3}{2}} w_f^{-1} \qquad (15.4)$$

$$\psi = g(\rho_s-\rho) d / \tau_0 \qquad (15.5)$$

in which w_f is a non-dimensional expression of the settling velocity and is given by:

$$w_f = \left(\frac{2}{3} + R_w^{-1}\right)^{\frac{1}{2}} - R_w^{-\frac{1}{2}} \qquad (15.6a)$$

$$R_w = \nu w_s / d \ , \ w_s \approx 2.4\, g\,(\rho_s/\rho - 1)\,\nu^{-1} d^2 \qquad (15.6b,c)$$

where R_w is a Reynolds number for the particle. In fact, he determined the relation between φ and ψ from various experiments as:

$$0.465\,\psi = \exp(-0.391\,\varphi) \qquad (15.7)$$

However, the data plotted on φ-ψ coordinates show large dispersion for smaller ψ, that is, for larger values of stress and transport rate, which are encountered in many of actual streams.

Cartwright (1959) applied Einstein's formula for Q_s to dynamics of formation of sand-waves by tidal currents. He expressed Q_s as a function of fluid velocity at the bed. With an assumption that the variations of the fluid velocity are sufficiently slow for the sand transport to adjust itself to the fluid velocity, the sand eroded per second from a portion of the bed due to the current $U(z) + \bar{u}(z) e^{ikx}$ becomes

$$\frac{d\,Q_s(U)}{dU}\; ik\bar{u}\,e^{ikx}\,\delta x \qquad (15.8)$$

where $\bar{u}(z)\,e^{ikx}$ is a perturbation of the basic current $U(z)$ caused by a small irregularity of the bed

$$\zeta = \xi(\;\;.t)\,e^{ikx}$$

This rate of erosion is equal to $(\partial\zeta/\partial t\;)\,\delta x$ and thus the most equation of continuity is derived $\partial\zeta/\partial t = Q_s'(U)\,ik\bar{u}\,e^{ikx}$ (15.9)

The viscous boundary layer theory of fluid motion near the bed yields the equation of

$$\bar{u}(z) = [\alpha \quad + i\beta \quad]\,\xi(t) \qquad (15.10)$$

where α and β are real coefficients depending on the basic current $U(z)$. If the basic current is tidal, its direction and speed change with time, but the transport Q_s becomes

prominent only near maximum ebb and flood. The amplitude spectrum of ξ can be determined from equation (15·10) as a function of $\overline{U}$, $\partial \overline{U}/\partial t$ and other quantities related to the perturbation velocity.

Kalinske (1947) assumed that the transport rate Q_s is proportional to the average speed of grains $\overline{v_g}$, times the average volumes of a grain $(\pi/6)\, d^3$ times the average number of the grains per unit area $p_a \left(\frac{\pi}{4} d^2\right)^{-1}$, in which p_a depends on the closeness of packing of the grains. Thus

$$Q_s = \frac{\pi}{6} d^3 \rho_s g\, \overline{v_g}\, p_a \left(\frac{\pi}{4} d^2\right)^{-1} = \frac{2}{3} p_a d \rho_s g\, \overline{v_g} \qquad (15{\cdot}11)$$

The value of $\overline{v_g}$ can be evaluated by averaging the instantaneous velocity of a grain v_g :

$$v_g = b\,(u - u_c) \qquad (15{\cdot}12)$$

where u and u_c are respectively instantaneous velocity of the fluid and the critical fluid velocity that will start motion of the grain and b is a numerical constant. The frequency distribution of u is assumed to be given by the normal error law:

$$F(u) = (2\pi\sigma)^{-1} \exp\left[-(u-\bar{u})^2/2\sigma^2\right] \qquad (15{\cdot}13)$$

where $\sigma^2 = \overline{(u-\bar{u})^2}$ and $\bar{u}$ in the mean velocity. The mean value of $\overline{v_g}$ will be

$$\overline{v_g} = b \int_{u_c}^{\infty} (u-u_c)\, F(u)\, du = b\left[\frac{\sigma}{\sqrt{2\pi}} e^{-s_c^2/2} + (\bar{u}-u_c)\, F(s_c)\right] \qquad (15{\cdot}14)$$

in which $s_c = (u - \bar{u})\,\sigma^{-1}$.

Since $\bar{u}$ and u_c are proportional to $\sqrt{\tau_0}$ and $\sqrt{\tau_c}$ respectively, the equation (15.11) can be expressed in a form

$$Q_s \left[(\tau_0/\rho)^{\frac{1}{2}} \rho_s g d\right]^{-1} = \Phi\,(\tau_0/\tau_c) \qquad (15{\cdot}15)$$

He determined the function Φ for an argument τ_0/τ_c only from experimental data and assuming that the τ_0 includes the effects of turbulent fluctuations σ implicitly.

The suspended-load equation currently used for determining concentration C in open channel is derived from the Fickian diffusion equation. In a steady state, this equation is written as:

$$c\,w_s = -\eta_s (dc/dz) \qquad (15\cdot16)$$

in which w_s is the settling velocity of a particle and η_s is the vertical eddy diffusivity for the suspended material. The mixing length theory is applied to this problem. The shearing stress, velocity distribution and eddy viscosity of the flow in the channel, respectively are given by:

$$\tau = \tau_0 (1 - z/h) \qquad (15\cdot17)$$

$$du/dz = u_* / kz \qquad (u_* = \sqrt{\tau_0/\rho}) \qquad (15\cdot18)$$

$$\eta = k_0 u_* z(1 - z/h) \qquad (15\cdot19)$$

where h is the depth and $z = 0$ at the bed. If it is assumed that $\eta_s = \alpha\,\eta$, the integration of equation (15.16) with z yields

$$C/C_a = \left[\frac{h-z}{h-a}\,\frac{a}{n}\right]^{w(\alpha k_0 u_*)^{-1}} \qquad (15\cdot20)$$

where a is a reference level above the bed at which the concentration C_a is known. The C_a was determined in terms of settling velocity and frequency distributions of vertical turbulent velocity by Lane and Kalinske (1939), integrating again the probability distribution of the vertical turbulent velocity that is greater than settling velocity.

Numerous laboratory and field measurements were made to test the validity of equation (15.20). These data indicate that the

values of α varies from 1.0 to 1.5, indicating that η_s tends to be larger than η . Brush (1962) made laboratory experiments for the purpose of testing the similarity between momentum transport and sediment transport, by measuring velocity and concentration distribution in a wake of a ¼-inch submerged vertical jet; which was directed into a large cylindrical tank. He found that the value of α is 0.15, 0.50 and 1.00 for particle size of 0.55, 0.38 and 0.19 mm, respectively, on the contrary to other experiments made in one-dimensional shear stream.

Davies (1952) discussed the suspension of sediments in a turbulent fluid using statistical theory of diffusion. In one dimensional diffusion, $f(\xi - x; \Delta t, \xi, t)$ is defined as the transition probability that a particle located at ξ at time t will go to x at time $t + \Delta t$. This function must satisfy the conditions:

$$f(\xi - x; \Delta t\ \xi, t) \rightarrow \delta(x - \xi) \quad \text{as}\ \Delta t \rightarrow 0 \qquad (15{\cdot}21)$$

$$\int_{-\infty}^{\infty} f(\xi - x; \Delta t, \xi, t)\, d(\xi - x) = 1 \qquad (15{\cdot}22)$$

Then Smoluchowski's classical diffusion theory is expressed by the integral equation

$$C(x, t+\Delta t) = \int_{-\infty}^{\infty} f(\xi - x; \Delta t, \xi, t)\, C(\xi, t)\, d(\xi - x). \qquad (15{\cdot}23)$$

The other transition probability $g(\xi - x; -\Delta t, \xi, t)$ that a particle located at ξ at time t came from x at time $t - \Delta t$. This function g satisfied the same conditions (15·21) and (15·22).

Then $C(x, t-\Delta t$ can be defined similarly to (15·23). If expansions:

$$C(x, t+\Delta t) = C(x,t) + C_t(x,t)\Delta t + C_{tt}(\Delta t^2/2!) + \cdots \tag{15·24}$$

$$f(\xi - x; \Delta t, \xi, t)\, C(\xi, t) = \sum_{n=0}^{\infty} \left(\frac{1}{n!}\right)\left\{(x-\xi)^n (\partial/\partial x)^n C(x,t)\, f(\xi - x; \Delta t, x, t)\right\} \tag{15·25}$$

are substituted, equation (15·23) can be written as:

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}(a_1 C) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(a_2 C) + \frac{1}{3!}\frac{\partial^3}{\partial x^3}(a_3 C) + \cdots \tag{15·26}$$

where

$$a_n = \lim_{\Delta t \to 0} \int \frac{(x-\xi)^n}{\Delta t} f(\xi - x; \Delta t, x, t)\, d(\xi - x) \tag{15·27}$$

The similar expressions are obtained for the equations related to g.

If the integral equation for $\frac{1}{2}\{C(x, t+\Delta t) + C(x, t-\Delta t)\}$ is substituted by expansions (15·24) and (15·25) and by those related to g, the differential equation becomes:

$$\frac{1}{2}\frac{\partial^2 C}{\partial t^2} = \frac{\partial}{\partial x}(\alpha_1 C) + \frac{1}{2!}\frac{\partial^2}{\partial x^2}(\alpha_2 C) + \frac{1}{3!}\frac{\partial^3}{\partial x^3}(\alpha_3 C) + \cdots \tag{15·28}$$

where:

$$\alpha_n(x, t) = \lim_{\Delta t \to 0} \int \frac{(x-\xi)^n}{\Delta t} \frac{1}{2}(f+g)\, d(\xi - x) \tag{15·29}$$

For a Browninan motion, a particle moves in jumps or discontinuously but if the limit $\Delta x/\Delta t$ exists as $\Delta t \to 0$, the physical process is called "diffusion". In the fluid motion, the particle move continuously with a speed $v_D = \lim_{\Delta t \to \infty} \Delta x/\Delta t$ and then alter its direction instantaneously. But if the limit $2\Delta x/\Delta t^2$ exists as $\Delta t \to 0$, the process is called "diffusion by continuous movement" and is applicable to the diffusion by turbulence.

The second moment α_2 is mean square velocity $\langle u^2 \rangle$. With the condition that Lagrangian derivatives of $\langle u^2 \rangle$ and C are zero, it is shown that the term $\langle u^2 \rangle_x C_x = \frac{\partial \langle u^2 \rangle}{\partial x} \frac{\partial C}{\partial x}$ in $\frac{\partial^2}{\partial x^2}(\alpha_2 C)$ of equation (15.22) is equal to:

$$- \frac{\langle u^2 \rangle_t}{\langle u^2 \rangle} C_t = - \lim_{\Delta t \to 0} \left\{ \frac{1 - R(\Delta t, t)}{\Delta t} \right\} C_t \qquad (15{\cdot}30)$$

where $R(\Delta t, t)$ is the velocity correlation function. The coefficient of C_t is the microscale of turbulence.

The equation (15.28) now becomes similar to Goldstein's equation (1951) which was discussed in Chapter 12 .

Davies, also derived the transport equation for suspended particles from energy consideration. The kinetic energy of the sand particles are equal to $\frac{1}{2}\rho_s \cdot \langle v_g^2 \rangle \cdot C(z)$ where v_g is the particle velocity. The particle pressure is twice this kinetic energy. Thus, the equation of motion of an aggregation of the particles in vertical direction becomes:

$$\frac{d}{dz}\left[\rho_s C(z) \langle v_g^2 \rangle\right] + \frac{2\rho_s(\rho_s - \rho)}{2\rho_s + \rho} g C(z) = 0 \qquad (15{\cdot}31)$$

where the second term represents the gravatational force on the particles.

In a steady state, the exchange of energy between the fluid and particles reaches equilibrium and the time average energy of both systems becomes equal to each other:

$$\rho_s \langle v_g^2 \rangle = \rho \langle u^2 \rangle = 2T(y) \qquad (15{\cdot}32)$$

The acceleration on a particle due to fluid viscosity is proportional to the kinematic viscosity ν, times a velocity difference between the fluid and particle. The average velocity difference is expressed in terms of $T(y)$ by

$$\left(\langle u^2 \rangle\right)^{\frac{1}{2}} - \left(\langle v_g^2 \rangle\right)^{\frac{1}{2}} = \left[1 - (\rho/\rho_s)^{\frac{1}{2}}\right]\left[T(y)/\rho\right]^{\frac{1}{2}} \qquad (15{\cdot}33)$$

The equation of motion (15·32) becomes

$$\nu\alpha \frac{d^2}{dz^2}\left[(T/\rho)^{\frac{1}{2}} C\right] + \frac{d}{dz}\left(TC/\rho_s\right) + \frac{(\rho_s-\rho)}{\rho+2\rho_s} gC = 0 \qquad (15{\cdot}34)$$

in which α is a dimensionless constant depending on the shape of particles and density differences between the fluid and particles.

The influence of the finite volume of the particles can be incorporated into the equation (15·34) taking analogy with the Van der Waals' equation of state:

$$PV = KT\left(1 - b/V\right)^{-1} - F \qquad (15{\cdot}35)$$

where V is the volume of the vessel, b is four times volume of the molecules, KT is the kinetic energy of the molecules and F is a term due to the attractive intermolecular forces. In the present problem, $(b/V) = 4C$, since V is taken as a unit volume. The function F is considered as equal to the gradient of the viscosity term derived before, and the pressure for the sands-fluid system becomes:

$$\frac{P}{\rho_s} = \frac{TC}{\rho_s}\,\frac{1}{(1-4C)} + \alpha\nu\frac{d}{dz}\left[\left(\frac{T}{\rho}\right)^{\frac{1}{2}} C\right] \qquad (15{\cdot}36)$$

Thus, the equation (15·34) is modified into:

$$\nu\alpha\frac{d^2}{dz^2}\left[(T/\rho)^{\frac{1}{2}} C\right] + \frac{d}{dz}\left[\frac{T}{\rho_s}\,\frac{C}{1-4C}\right] + \left(\frac{\rho_s-\rho}{\rho+2\rho_s}\right) gC = 0 \qquad (15{\cdot}37)$$

In a non-stationary condition the classical Fickian equation of sedimentation is: $\frac{\partial C}{\partial t} - w_s\frac{\partial C}{\partial z} = \frac{\partial}{\partial z}\left(\eta_s\frac{\partial C}{\partial z}\right)$ (15·38)

in which w_s is the settling velocity. The generalised two dimensional diffusion equation which is based on the equation (15·28) and (15·31) becomes:

$$C_{tt} = \left(\frac{\rho_s-\rho}{2\rho_s+\rho}\right) g C_z + \left[\langle u^2\rangle C\right]_{xx} + \left[\langle w^2\rangle C\right]_{zz}$$

where:

$$+ 2\left[\langle uw\rangle C\right]_{xz} + \frac{\nu}{3!}\left\{\left[\left(\langle u^2\rangle + \langle w^2\rangle\right)\right]^{\frac{3}{2}} C\right\}_{xzx} + \cdots \qquad (15{\cdot}39)$$

$$\begin{bmatrix} \langle u^2\rangle & \langle uw\rangle \\ \langle wu\rangle & \langle w^2\rangle \end{bmatrix}$$

is the energy tensor or stress tensor of the particles. This equation is a parabolic partial differential equation. If the third and higher order derivatives are neglected it becomes hyperbolic. This means that a source of particles released at the origin has an effect on the entire turbulent field immediately, but the main part moves like a wave with decaying tail following the wave front.

The theory of heavy particle diffusion in the light of statistical theory of turbulent was discussed by Yudine (1959) and Smith (1959) in their study of diffusion of particles suspended in the atmosphere. Their arguments included neither boundaries nor shear flows unlike the problem of sediment diffusion. However, some of their results may be useful for further study in the sedimentation process.

Yudine considered the velocity correlation function between vertical velocities of a falling particle at time t and $t+\lambda$

$$\overline{w(t,z)\,w(t+\lambda, z-w_s\lambda+\delta)} = B(\lambda,\zeta) \qquad (15.40)$$

This equation is based on four assumptions: (1) the time elapsed from the beginning of the process satisfies the condition

$$t \gg w_s/g$$

so that the effect of inertia may be neglected, (2) there is no difference between the velocity fluctuations of the medium and those of falling particle, (3) the particle at z at time t will be found at $z - w_s\lambda + \delta$, where δ is the integrated distance due to fluctuations of velocity w and w_s is the settling velocity, (4) that the fluid turbulence is homogeneous.

In the inertial subrange of locally isotropic turbulence, the form of $B(\lambda,\xi)$ for $\lambda=0$ or $\xi=0$ is known, i.e.

$$B(\lambda,0) = B_0 - aD|\lambda| \qquad (15\cdot41)$$

where D is the dissipation of energy, a is a numerical factor of order unity, and

$$B(0,\xi) = B_0 - b\,D^{2/3}|\xi|^{2/3} \qquad (15\cdot42)$$

where b is another numerical factor of order unity. In these equations $2aD\lambda$ and $2b\,D^{2/3}\xi^{2/3}$ are the time and space structure function, respectively.

For heavy particles, $B(\lambda,0)$ or $B(0,\xi)$ is assumed to be zero beyond the limit of $|\lambda|$ or $|\xi|$, for which relation (15·41) or (15·42) vanishes. Introducing new variables

$$p = aD\lambda/B_0 \;;\quad q = b(D\xi)^{2/3}/B_0 \qquad (15\cdot43)$$

the function $B(\lambda,\xi)$ can be expressed as a function of p and q, $Z(p,q)$. The lower limit and upper limit function for $Z(p,q)$, satisfying relations (15·41) and (15·42), are given by:

$$\begin{aligned} Z_1(p,q) &= B_0(1-|p|-|q|) && \text{for } |p|+|q| \leq 1 \\ &= 0 && \text{for } |p|+|q| \geqslant 0 \qquad (15\cdot44) \end{aligned}$$

and

$$\begin{aligned} Z_2(p,q) &= B_0(1-|p|) && \text{for } |q| \leq |p| \leq 1 \\ &= B_0(1-|q|) && \text{for } |p| \leq |q| \leq 1 \\ &= 0 && \text{for } |p| \geqslant 1 \text{ or } |q| > 1 \qquad (15\cdot45) \end{aligned}$$

respectively, for the domain $p > 0$ and $q > 0$. The diffusion coefficient η is determined by:

$$\eta = \int_0^{\infty} B(\lambda, w_s\lambda)\,d\lambda \qquad (15\cdot46)$$

and substitution of (15.44) and (15.45) into (15.46) yields the lower and upper limit of η as

$$\eta_1 = \frac{B_0^2}{aD} \left(0.4\, m^3 + 0.1\, m^6\right) \qquad (15.47a)$$

and

$$\eta_2 = \frac{B_0^2}{aD} \left(0.5 - 0.1\, \mu^4\right) \quad \text{when } \mu \leq 1$$
$$= \frac{B_0^2}{aD} \times \left(0.4\, \mu^{-1}\right. \quad \text{when } \mu \gg 1 \qquad (15.47b)$$

where m is the solution of the cubic equation

$$m^3 + \mu^{2/3} m^2 - 1 = 0 \qquad (15.48a)$$

and the parameter μ has the value

$$\mu = \delta^{3/2} a^{-1} w_\Delta B_0^{-1/2} \qquad (15.48b)$$

The difference in values of η_2 and η_1 is zero at $\mu = 0$ reach maximum at $\mu = 1$ and again almost negligible for $\mu \gg 5$.

Smith (1959) treated the spread of particles falling with a sufficiently large settling velocity so that the eddy structure remains sensibly unchanged while it affects the cluster. This condition is satisfied when the Eulerian time-scale for the falling-cluster l_σ / w_δ is much bigger than the Lagrangian time scale t_L $(\approx l[\overline{(u')^2}]^{-\frac{1}{2}})$, where l_σ is the length-scale of the cluster and l is a measure of the energy-containing eddies.

The velocity spectrum of particles whose distribution is Gaussian with standard deviation σ (equivalent to cluster size) is derived, by considering the deviations of the particle velocity from the average in the cluster V. Let $u_i(\xi)$ be the velocity in the i-th direction at a point $\vec{\xi}$ from the center $\vec{r}$ of the cluster, $P(\vec{\xi})$ be the probability distribution of particles in V. Then the average velocity in V is: $\langle u_i(\vec{r})\rangle_V = \int_V u_i(\vec{\xi}+\vec{r}) P(\vec{\xi})\, d\vec{\xi}$

(15.49)

and the deviations from this average

$$u_i'(\vec{\xi}+\vec{r}) = u_i(\vec{\xi}+\vec{r}) - \langle u_i(\vec{r})\rangle \qquad (15.50)$$

The average of $u_i'^2$ for all realizations becomes:

$$\overline{u_i'^2} = \lim_{A\to\infty}\frac{1}{A}\int_A d\vec{r}\int_V u_i'^2(\vec{\xi}+\vec{r})\,P(\vec{\xi})\,d\vec{\xi} \qquad (15.51)$$

Substituting (15.49) and (15.50) into (15.51) yields

$$\begin{aligned}\overline{u_i'^2} &= \int_V P(\vec{\xi})\,d\vec{\xi}\lim_{A\to\infty}\frac{1}{A}\int_A u^2(\vec{\xi}+\vec{r})\,d\vec{r} - \lim_{A\to\infty}\frac{1}{A}\int_A d\vec{r}\Big\{\int_V u_i(\vec{\xi}+\vec{r})\cdot\\ &\quad\cdot P(\vec{\xi})\,d\vec{\xi}\Big\}\times\int_V u_i(\vec{\xi}+\vec{r})\,P(\vec{\xi})\,d\vec{\xi}\\ &= \overline{u_i^2} - \iint_{VV} P(\vec{\xi})\,P(\vec{\xi})\,R_{ii}(\vec{\xi}-\vec{\xi})\,d\vec{\xi}\,d\vec{\xi} \qquad (15.52)\end{aligned}$$

$$\overline{u_i'^2} = \overline{u_i^2} - \frac{1}{2\sqrt{\pi\sigma^3}}\int_0^\infty \lambda^2 R_{ii}(\lambda)\,e^{-(1/4\sigma^2)\lambda^2}\,d\lambda \qquad (15.53)$$

The second relation is obtained from the assumption of a Gaussian cluster in isotropic turbulence.

The expression of $R_{ii}(\lambda)$ in terms of the energy density $E(k)$:

$$R_{ii}(\lambda) = 2\int_0^\infty E(k)\,\frac{\sin k\lambda}{k\lambda}\,dk \qquad (15.54)$$

is substituted in (15.53), resulting

$$\frac{1}{2}\overline{u_i'^2} = \frac{1}{2}\overline{u_i^2}\int_0^\infty E(k)\,(1-e^{-\sigma^2k^2})\,dk = \frac{1}{2}\overline{u_i^2}\int_0^\infty E_\sigma(k)\,dk \qquad (15.55)$$

where, $E_\sigma(k) = E(k)\,(1-e^{-\sigma^2k^2})$. The factor $(1-e^{-\sigma^2k^2})$ is a measure of the response of a cluster to the energy in wave number k. The function $E_\sigma(k)$ is the modified spectrum and the value of $(k_m)^{-1}$ at which $E_\sigma(k)$ attains its maximum is a measure of the length-scale l_σ of turbulence as experienced by the cluster.

In order to determine the cluster size σ, the separation of two particles, say rth and sth particles, $\vec{\xi} = (\xi_i)$ is expressed in terms of their respective velocities u_{ri} and u_{si}:

$$\frac{d\xi_i}{dt} = u_{ri} - u_{si} \qquad (15.56)$$

Combining this with an integrated form with t, we have

$$d\xi_i^2 = 2\int[u_{ri}(t) - u_{si}(t)][u_{ri}(\lambda) - u_{si}(\lambda)]\,d\lambda \qquad (15.57)$$

where the repeated index means summation over i. The value of w_s is so large that correlations of velocities of particles in the same cluster but at different times become equivalent to those of velocities of particles in two identical clusters at the same instant. Therefore,

$$\overline{u_{ri}(t)\, u_{si}(\lambda)} = \overline{u_i(\vec{x})\, u_i(\vec{x}+\vec{z}+\vec{\xi})} \qquad (15{\cdot}58)$$

where $\vec{z}$ is a vertical vector of magnitude $w_s(t-\lambda)$.

Taking average of (15.56) over all realisations and evaluating the velocity correlations, we have:

$$d\,\overline{\xi^2}/dt = \frac{4}{w_s}\int_0^{w_s t}\left[R_{ii}(|z|) - \overline{R_{ii}(z+\xi)}\right]dz \qquad (15{\cdot}59)$$

in which the bar over $R_{ii}(z+\xi)$ means averaging all pairs of particles. The Gaussian distribution of particles yields the average of the separation ξ by

$$\overline{\xi^2} = 6\,\sigma^2 \qquad (15{\cdot}60)$$

For the specific form of correlation functions

$$R_{11}(r,0,0) = e^{-r/l} \qquad (15{\cdot}61)$$

equation (15.59) can be expressed in a form

$$d\,\sigma^2/dt = \frac{4}{3w_s}\,\frac{\overline{u^2}}{l}\,\sigma\, I(\sigma, t) \qquad (15{\cdot}62)$$

The spreading of the cluster size σ has three different stages.

For very small times, $\sigma = \sigma(0) + \frac{8}{3\sqrt{\pi}}\frac{\overline{u^2}}{l}t^2$ (15.63a)

For intermediate times, $\sigma = \sigma(0)\exp\left[2\overline{u^2}(w_s l)^{-1} Ht\right]$ (15.63b)

where $H = H(w_s t/3\sigma(0))$ is a function incrdasing from 0 to 0.85 ca. as the argument increases from 0 to 5. For large times satisfying

$$w_s t > 2l \quad , \qquad \sigma^2 \approx \frac{4}{3}\,\overline{u^2}\,l\,(w_s)^{-1}\,t \qquad (15{\cdot}63c)$$

which is similar to the relation derived from the Fickian diffusion.

Both Yudine's and Smith's approaches will become a starting point for treating sediment diffusion in the light of statistical and spectrum theory of turbulence, although modification is

necessary by taking into account the effects of bottom and the turbulence structure in the shear flow.

Hunt (1954) discussed the dynamics of the system of mixture of water and sediments. The flux vector of sediment and water was defined respectively by $\vec{P} = \vec{u}C - \eta_s \nabla C$ and $\vec{Q} = \vec{v}(1-C) - \eta_w (1-C)$, where the suffices s and w refer to the quantities related to sediment and water, respectively. The equation of the rate of change of sediment concentration is respectively given by:

$$\nabla \cdot (\vec{P} + \vec{Q}) = 0 \;; \qquad \frac{\partial C}{\partial t} = -\nabla \cdot \vec{P} \qquad (15.64\,a,b)$$

In the case of steady uniform flow where the concentration varies only in the vertical direction z , only the terms of vertical advection and diffusion remains. The vertical velocity of the sediments u_z is equal to $v_z - w_s$, where w_s is the settling velocity. Elimination of u_z and v_z between (15.64a) and (15.64b) yields

$$\eta_s \frac{\partial C}{\partial z} + C \frac{\partial C}{\partial z} (\eta_w - \eta_s) + (1-C) C w_s = 0 \qquad (15.65)$$

Where there are more than one component in the material in suspension, the flux vector for each component is defined as $\vec{P}_n = \vec{u}_n C_n - \eta_s \nabla C_n$ and the continuity condition becomes

$$\nabla \cdot \Big(\sum_n \vec{P}_n + \vec{Q} \Big) = 0 \qquad (15.66)$$

For the steady uniform flow, the equation for concentration of each component becomes

$$\eta_s \frac{\partial C_n}{\partial z}(1-C) + C_n\, w_{s,n} \frac{\partial}{\partial z} C + C_n w_{s,n} (1-C) = 0 \qquad (15.67)$$

where

$$C = \sum_n C_n$$

The mixing length theory with Karman's similarity hypothesis

yields the velocity distribution in an open channel of the depth h:

$$\frac{U_m - U}{(ghS)^{\frac{1}{2}}} = -\frac{1}{k_o}\left[(1-z/h)^{\frac{1}{2}} + B_H \ln\left\{1-(1-z/h)^{\frac{1}{2}}B_H^{-1}\right\}\right] \quad (15.68)$$

and the shearing stress

$$\tau = \tau_0 (1 - z/h) \quad ; \quad \tau_0 = \rho g h S \quad (15.69\,a,b)$$

where S is the energy gradient, B_H is a constant and U_m is the maximum velocity encountered at the surface $z = h$.

The eddy viscosity is given by:

$$\eta = \tau\left[\rho \frac{dU}{dz}\right]^{-1} = 2k_o h\,(ghS)^{\frac{1}{2}}(1-z/h)\left[B_H - (1-z/h)^{\frac{1}{2}}\right] \quad (15.70)$$

Substituting this value into equation (15.65), in which the second term of the left side is neglected owing to its smallness, we have

$$\left(\frac{C_1}{1-C_1}\right)\left(\frac{1-C_a}{C_a}\right) = \left(\frac{1-z/h}{1-a/h}\right)^{\frac{1}{2}}\left\{\frac{B_H-(1-a/h)^{\frac{1}{2}}}{B_H-(1-z/h)^{\frac{1}{2}}}\right\}^{s} \quad (15.71)$$

where $s = w_s (k_o B_H)^{-1} (ghS)^{-\frac{1}{2}}$

and C_a is the concentration at a reference level $z = a$. Hunt tested this relation by comparing with the observed distribution of suspended matter in an experimental flume.

16. Sound transmission in a turbulent medium.

The velocity of propagation of sound waves in the sea depends on density and to lesser degrees on current. Since there is always some sort of inhomogeneity in temperature and salinity in the ocean caused by turbulence and convection, the sound wave is scattered by such inhomogeneity and shows random fluctuations in intensities, directions and phases. The scattering of underwater sound is a subject of military interest and a number of experimental and theoretical papers have appeared since World War II. On the other hand, the scattering is mainly caused by temperature microstructure due to small scale turbulence and thus measurements of sound scattering will give us information on structure of the micro turbulence, which would be difficult to be measured by other method.

The basic theory of scattering in the ray theory range was discussed by Bergmann (1946). The treatment from the wave theory was done by Pekeris (1947). Liebermann (1951) discussed the sound scattering measurements in the sea in the light of these theories, considering probable models of temperature inhomogeneity in the ocean. Mintzer (1953, 1954) developed the work of Pekeris and derived a rigorous expression for the first order space average value of the scattered intensity. Obukhov (1953) obtained a general expression for the fluctuation of phase and amplitude. This expression is equivalent to Mintzer's result in the ray-theory range. Skudrzyk (1957) discussed the scattering in an inhomogeneous medium in general, reviewing the existing theories.

The scattering of plane waves by a turbulent flow with temperature inhomogeneity is discussod by means of the wave theory. The spectrum functions of turbulence of the medium and those of random distributions of temperature will be incorporated in the intensity of scattered sound wave (Tatarskii, 1959).

The basic equation for sound propagation in a moving medium is

$$\nabla^2 P - C^{-2}\left(\frac{\partial}{\partial t} + u_i \frac{\partial}{\partial x_i}\right)^2 P = 0 \qquad (16\cdot1)$$

where P is the potential of sound motion, u_i are the velocity of the medium and C is the sound velocity. It is assumed that the mean velocity is zero and that the second or higher power terms of $|\vec{u}'|/C$ are negligible, where $\vec{u}'$ is the instantaneous velocity of the medium. Further, because the frequency of turbulent velocity of the fluid is much smaller than that of sound wave, equation (16·1) becomes

$$\nabla^2 P - C^{-2}\frac{\partial^2 P}{\partial t^2} = 2\vec{u}'\, C^{-2}\cdot\nabla\frac{\partial P}{\partial t} \qquad (16\cdot2)$$

with an accuracy up to terms of order $|\vec{u}'|/C$

In the atmosphere and the ocean the sound velocity C is a function of temperature θ like $\Theta^{\frac{1}{2}}$. Therefore, if $\bar{\theta}$ and θ' denote the mean and fluctuation temperature, respectively, we have

$$C(\theta) \sim C(\bar{\theta})\left(1+\tfrac{1}{2}\theta'/\bar{\theta}\right) \qquad (16\cdot3)$$

Then equation (16.2) becomes

$$\nabla^2 P - \frac{1}{c^2}\frac{\partial^2 P}{\partial t^2} = \frac{2\vec{u}'}{c^2}\cdot\nabla\frac{\partial P}{\partial t} - \frac{1}{c^2}\frac{\theta'}{\bar{\theta}}\frac{\partial^2 P}{\partial t^2} \qquad (16.4)$$

with an accuracy up to terms of order $\theta'/\bar{\theta}$ which is of the same order as u'/c in the atmosphere. In the ocean, the ratio of the first to the second term of the right side is almost equal to $2(u'/c)(\theta'/\bar{\theta})^{-1}$ and thus, the first term can be neglected.

Assuming that $P = \Pi e^{-i\omega t}$ with $\Pi = \Pi_0 + \Pi_1 + \Pi_2 + \cdots$ where Π_n is of the order of $|u'/c|^n$ or $(\theta'/\bar{\theta})^n$, equation (16.5) can be divided into

$$\nabla^2\Pi_0 + k^2\Pi_0 = 0 \qquad (16.5)$$

$$\nabla^2\Pi_1 + k^2\Pi_1 = -2ik\frac{\vec{u}'}{c}\cdot\nabla\Pi_0 + k^2\frac{\theta'}{\bar{\theta}}\Pi_0 \qquad (16.6)$$

where $k = \omega/c$ is the wave number.

The zero-th approximation Π_0 represents the incident wave and for the plane wave it is written as

$$\Pi_0 = A_0 e^{i\vec{k}\cdot\vec{r}} \qquad (16.7)$$

where $\vec{k}$ is the wave number vector. Substituting this into (16.6), we have

$$\nabla^2\Pi_1 + k^2\Pi_1 = 2k^2\left(\frac{\vec{u}'\cdot\vec{n}}{c} + \frac{\theta'}{2\bar{\theta}}\right)A_0 e^{i\vec{k}\cdot\vec{r}} \qquad (16.8)$$

where $\vec{n}$ is a unit vector in the direction of $\vec{k}$.

The solution of the Poisson equation

$$\nabla^2\phi + k^2\phi = f(\vec{r}) \qquad (16.9)$$

corresponding to outgoing waves is of the form

$$\phi(\vec{r}) = -\frac{1}{4\pi}\int_V f(\vec{r}')e^{ik|\vec{r}-\vec{r}'|}|\vec{r}-\vec{r}'|^{-1}dV' \qquad (16.10)$$

where $\vec{r}'$ is a position vector ranging over the scattering volume and the origin of the coordinate is taken inside V. If $|\vec{r}|$ is much larger than the dimension of V, the quantity $|\vec{r}-\vec{r}'|$ can be expanded in a series of powers of r'/r, i.e.

$$|\vec{r}-\vec{r}'| = r - \vec{m}\cdot\vec{r}' + (2r)^{-1}[(r')^2 - (\vec{m}\cdot\vec{r}')^2] + \cdots \qquad (16\cdot11)$$

where $\vec{m} = \vec{r}/r$ is a unit vector in the direction $\vec{r}$. If

$$(2r)^{-1}[(r')^2 - (\vec{m}\cdot\vec{r}')^2] \ll 1 \qquad (16\cdot12)$$

for all $\vec{r}'$, i.e. if $\lambda r \gg L^2$, where L and λ are the dimension of the volume V and wave length, respectively, then equation (16·10) becomes

$$\phi(\vec{r}) \approx Q\frac{e^{ikr}}{r} \qquad (16\cdot13a)$$

$$Q = \frac{1}{4\pi}\int_V f(\vec{r}')e^{-ik\vec{m}\cdot\vec{r}'}\,dV' \qquad (16\cdot13b)$$

where $|\vec{r}-\vec{r}'|$ of the denominator of (16·10) is replaced by r. This expression represents a spherical wave. The function $\phi(\vec{r})$ and $f(\vec{r}')$ of equation (16·13) can be replaced by $\Pi_1(\vec{r})$ and the right hand side of equation (16·8).

The average value of the flux density vector of the scattered energy is equal to

$$\vec{S} = \frac{1}{2}\omega\rho\; I_m\left(\Pi_1^{*}\nabla\Pi_1\right) \qquad (16\cdot14)$$

where I_m and the $*$ mean the imaginary part and conjugate value, respectively.

Assuming $kr \gg 1$, we have

$$\nabla \Pi_1 = Q\left(ik\frac{e^{ikr}}{r} - \frac{e^{ikr}}{r^2}\right)\vec{m} \approx ikQ\frac{e^{ikr}}{r}\vec{m} \qquad (16\cdot15)$$

Thus

$$\vec{S} = \frac{1}{2}\omega\rho k Q Q^{*}\vec{m} \qquad (16\cdot16)$$

The amplitude Q is random due to the random fluctuations of velocities and temperature. The mean value of $\vec{S}$ equals

$$\overline{\vec{S}} = \frac{1}{2}\frac{\omega\rho k}{r^2}\overline{QQ^{*}}\,\vec{m} = \vec{m}\frac{\rho c k^6}{8\pi^2 r^2}A_0^2 \times$$

$$\int_V\int_V \left[\frac{n_i u_i'(\vec{r}_1)}{c} + \frac{\theta'(\vec{r}_1)}{2\bar{\theta}}\right]\left[\frac{n_j u_j'(\vec{r}_2)}{c} + \frac{\theta'(\vec{r}_2)}{2\bar{\theta}}\right]$$

$$\times\, e^{ik(\vec{n}-\vec{m})(\vec{r}_1-\vec{r}_2)}\,dv_1\,dv_2 \qquad (16\cdot17)$$

With the assumption that $\vec{u}'(\vec{r})$ and $\theta'(\vec{r})$ are homogeneous and isotropic, the correlation functions are written as:

$$\overline{u_i'(\vec{r}_1)u_j(\vec{r}_2)} = B_{ij}(\vec{r}_1-\vec{r}_2) \qquad (16\cdot18a)$$

$$\overline{\theta'(\vec{r}_1)\theta'(\vec{r}_2)} = B_\theta(\vec{r}_1-\vec{r}_2) \qquad (16\cdot18b)$$

because they depend only on $\vec{r}_1 - \vec{r}_2$. The condition of incompressibility leads to

$$\overline{n_i u_i'(\vec{r}_1)\theta'(\vec{r}_2)} = 0 \qquad (16\cdot19)$$

Thus, the double integral of equation (16·17) is reduced to one volume integral, i.e.

$$\vec{S} = \vec{m}\,\frac{\rho c k^6 A_0 V}{8\pi^2 r^2}\Big[\frac{1}{c^2} n_i n_j \int_V B_{ij}(\vec{r}\,')\,e^{ik(\vec{n}-\vec{m})\vec{r}\,'}\,dV' + \frac{1}{4\theta^2}\int_V B_\theta(\vec{r}\,')\,e^{ik(\vec{n}-\vec{m})\cdot\vec{r}\,'}\,dV'\Big] \qquad (16{\cdot}20)$$

The correlation function $B_n(\vec{r})$ which is included in the integral of equation (16·20) can be expressed by a Fourier integral

$$B_n(\vec{r}) = \iiint_{-\infty}^{\infty} e^{-i\vec{\varkappa}\cdot\vec{r}}\,\Phi_n(\vec{\varkappa})\,d\vec{\varkappa} \qquad (16{\cdot}21)$$

where $\Phi_n(\vec{\varkappa})$ is a spectral density function. Substituting this in the integral

$$I = \frac{1}{8\pi^3}\int_V B_n(\vec{r})\,e^{i[(\vec{k}-k\vec{m})\cdot\vec{r}]}\,dV \qquad (16{\cdot}22)$$

we have

$$I = \frac{1}{8\pi^3}\iiint_{-\infty}^{\infty} \Phi_n(\vec{\varkappa})\,d\vec{\varkappa}\int_V e^{i\vec{\lambda}\cdot\vec{r}}\,dV$$

$$= \iiint_{-\infty}^{\infty} \Phi_n(\vec{\varkappa})\,F(\vec{\lambda})\,d\vec{\varkappa} \qquad (16{\cdot}23a)$$

$$F(\vec{\lambda}) = \frac{1}{8\pi^3}\int_V e^{i\vec{\lambda}\cdot\vec{r}}\,dV \qquad (16{\cdot}23b)$$

where $\vec{\lambda} = \vec{k} - k\vec{m} - \vec{\varkappa}$. The function $F(\vec{\lambda})$ equals $\delta(\vec{\lambda})$ if the volume V is infinite, and therefore $I = \Phi_n(\vec{k} - k\vec{m})$.

In the case of a finite volume of V, the function has a sharp maximum near $\lambda = 0$ and oscillates and falls off

rapidly as λ increases, while

$$\iiint_{-\infty}^{\infty} F(\vec{\lambda})\,d\vec{\lambda} = \int_{V} \delta(\vec{r})\,dV = 1 \qquad (16{\cdot}24)$$

and $F(0) = V(\delta\pi^3)^{-1}$. Therefore, $F(\vec{\lambda})$ is appreciably different from zero only in a region of the wave vector space with a volume of order $\delta\pi^3/V$. Thus,

$$I \approx \iiint_{V_L} \Phi_n(\vec{\chi})\frac{V}{\delta\pi^3}\,d\vec{\chi} = \widetilde{\Phi}_n(\vec{k} - k\vec{m}) \qquad (16{\cdot}25)$$

where V_L represent the region in space $\vec{\chi}$ with volume $8\pi^3/V$ near the point $\vec{\chi} = \vec{k} - k\vec{m}$, and $\widetilde{\Phi}_n(\vec{k} - k\vec{m})$ is the average of $\Phi(\vec{\chi})$ over the volume V_L.

The spectral density of temperature is denoted by $\Phi_\theta(\vec{\chi})$ Let $E(\vec{\chi})$ be the spectral density of turbulence energy. Then

$$B_{ij}(\vec{r}) = \iiint_{-\infty}^{\infty} e^{i\vec{\chi}\vec{r}}\left(\delta_{ij} - \frac{\chi_i\chi_j}{\chi^2}\right)E(\vec{\chi})\,d\vec{\chi} \qquad (16{\cdot}26)$$

Since two integrals of equation (16·20) have the same form as the integral I, they can be expressed by spectral densities $\widetilde{E}(\vec{\chi})$ and $\widetilde{\Phi}_\theta(\vec{\chi})$ as in equation (16·25). If the volume V is so large that the volume V_L with an order of magnitude $8\pi^3/V$ becomes small, the factor of $E\{k(\vec{n}-\vec{m})\}$ in the integral of equation (16.20) after substituting (16.26) can be simplified:

$$n_i n_k\left(\delta_{ik} - \frac{(n_i - m_i)(n_j - m_j)}{|\vec{n} - \vec{m}|^2}\right) = \frac{1}{2}(1 + \vec{m}\cdot\vec{n}) = \frac{1}{2}(1 + \cos\varphi) \qquad (16{\cdot}27)$$

where φ is the angle between the direction of the vector $\vec{k}$ and $\vec{n}$, i.e. the scattering angle. Therefore, we have

$$\vec{S} = \frac{\vec{m}}{R^2}\pi\rho c\, k^6 A_0^2 V\left[\frac{1}{c^2}E\{k(\vec{n}-\vec{m})\}\cos^2\frac{\varphi}{2} + \frac{1}{4\bar{\theta}^2}\Phi_\theta\{k(\vec{n}-\vec{m})\}\right] \qquad (16{\cdot}28)$$

In the case of isotropic turbulence $E(\vec{x}) = E(x)$ and $\Phi_\Theta(\vec{x}) = \Phi_\theta(x)$ and thus $k(\vec{n} - \vec{m}) = 2k\sin\frac{\varphi}{2}$. The energy flux density of the incident wave is equal to

$$\vec{S}_0 = \frac{1}{2}\omega\rho\, I_m\left(A_0 e^{-i\vec{k}\cdot\vec{r}} \cdot i\vec{k}\cdot A_0 e^{i\vec{k}\cdot\vec{r}}\right) = \frac{1}{2}\omega\rho A_0^2\vec{k} \qquad (16\cdot29)$$

The effective cross section $d\sigma(\varphi)$ for the scattering of sound in the direction φ is defined as ratio of scattered energy into the solid angle 2Ω, $S r^2 d\Omega$, to the absolute value of energy flux density of the incident wave. Thus

$$d\sigma(\varphi) = 2\pi k^4 V\left[\frac{1}{c^2}E\left(2k\sin\frac{\varphi}{2}\right)\cos^2\frac{\varphi}{2} + \frac{1}{4\pi^2}\Phi_\theta\left(2k\sin\frac{\varphi}{2}\right)\right] \qquad (16\cdot30)$$

This equation indicates that $\sigma(\varphi)$ depends only on spectral components of the turbulence with wave numbers $2k\sin\frac{\varphi}{2} = 2\pi/l(\varphi)$. If $l(\varphi) \ll L_0$, where L_0 is the upper limit of the isotropic domain of the turbulence, the spectral functions depend only on the local isotropic eddies and not on the large scale anisotropic eddies. If, further, $l_0 \ll l(\varphi) \ll L_0$ or

$$\lambda/L_0 \ll 2\sin\frac{\varphi}{2} \ll \lambda/l_0 \qquad (16\cdot31)$$

where $\lambda = 2\pi/k$ is wave length, velocity and temperature correlations, D_{rr} and D_θ, are expressed by Kolmogoroff's 2/3-power law:

$$D_{rr} = C_V^2 r^{2/3} \;;\quad D_\theta = C_\theta^2 r^{2/3} \qquad (16\cdot32\,a,b)$$

The spectral densities are given by:

$$E(\varkappa) = 0.061 C_V^2 \varkappa^{-11/3} \qquad (16.33a)$$

$$\Phi_\theta(\varkappa) = 0.033 C_\theta^2 \varkappa^{-11/3} \qquad (16.33b)$$

where $C_V^2 = a_1 \varepsilon^{2/3}$ $\quad C_\theta^2 = a_2 \bar{N} \varepsilon^{-1/3}$

ε is the energy dissipation rate of turbulence and $\bar{N}$ is the rate of dissipation of temperature inhomogeneities. (See Chapter 13). Then we have

$$d\sigma(\varphi) = 0.030 k^{1/3} V \left[\frac{C_V^2}{C^2} \cos^2\frac{\varphi}{2} + 0.13 \frac{C_\theta^2}{\bar{\theta}^2} \right] \left(\sin\frac{\varphi}{2}\right)^{-11/3} d\Omega \qquad (16.34)$$

Pekeris (1947) and Liebermann (1951) used the correlation function for temperature fluctuations: $R(r) = e^{-(r/a)}$ (16.35a) and Potter and Murphy (1955) used mathematically more rigorous function ; $R(r) = e^{-(r/a)^2}$ (16.35b)

Since in the sea the effects of turbulence velocity is negligible, the effective cross section of (16.30) depends on temperature correlation only. The ratio of the scattered intensity to the incident intensity is $|P_{sc}^2/P_i^2| = \frac{2\alpha^2 k^2 a^3}{\pi r^2} \left[1 + 4k^2 a^2 \sin^2\left(\frac{\varphi}{2}\right)\right]^{-1} dV$ (16.36a) for the correlation function (16.35a) and

$$|P_{sc}^2/P_i^2| = \frac{\alpha^2 k^4 a^3}{4\sqrt{\pi}\, r^2} \exp\left(-k^2 a^2 \sin^2\frac{\varphi}{2}\right) dV \qquad (16.36b)$$

for the function (16.35b), where $\alpha^2 = \overline{(\Delta C/C)^2}$ in which C and ΔC are sound velocity and its fluctuations, respectively, suffix i and sc means incident and scattered wave, respectively, and ΔV is an elementary volume of the scatterers.

The angular width $2\varphi_m$ of the scattered beam can be defined by $\pi\varphi_m^2 = \int_{-\pi/2}^{\pi/2} M(\varphi)\,d\varphi$, where $M(\varphi)$ is the directivity factor of (16·36a,b). The exponent and the Gaussian $R(r)$ respectively yield

$$\varphi_m \approx 1/ka \quad \text{and} \quad \varphi_m \approx 2/ka\,. \qquad (16{\cdot}37a,b)$$

The total energy I_{sc} scattered in the forward direction $0 \leq \varphi \leq \pi/2$ is given by integrating respectively equation (16·36a) or (16·36b):

$$I_{sc} \approx 2\alpha^2 k^2 a r I_i \qquad (16{\cdot}38a)$$

or

$$I_{sc} \approx \sqrt{\pi}\,\alpha^2 k^2 a r I_i \qquad (16{\cdot}38b)$$

The second expression is identical with Mintzer's (1954) result. The first equation can be interpreted as the total scattering by the aggregation of irregular patches whose average radius is a and the number is $n = r/a$. Since the scattering by a single patch is equal to $4\alpha^2 k^2 a^2 I_i$, the total scattering becomes

$$I_{sc} = (4\cdot\alpha^2 k^2 a^2 I_i)n = 4\alpha^2 k^2 a^2 \left(\frac{r}{a}\right) I_i \qquad (16{\cdot}39)$$

The above discussion is valid for scattering for large values of kr. For small values of kr the phase angle ϕ between P_{sc} and P_i becomes important. The scattering in such case was treated by Bergman (1946) and Lieberman (1951) from the theory of geometrical optics. Skudrzyk (1957) treated this case by an approximate method using a model of scattering by small patches.

If $|P_{sc}| \ll |P_i|$, the magnitude of the resulting pressure is given by

$$P = \left[(P_i + P_{sc}\cos\phi)^2 + (P_{sc}\sin\phi)^2\right]^{\frac{1}{2}} \approx P_i + P_{sc}\cos\phi + \cdots \qquad (16{\cdot}40)$$

The scattering by small patches causes the phase angle $\pm\pi/2 + \Delta\phi$ as the first approximation, where $\Delta\phi$ is the phase delay due to the difference in path between the scattered and the direct rays. Since the maximum opening of the beams scattered at individual patches is equal to $\gamma\varphi_m$ where φ_m is given by equation (16·37a) and γ is a constant nearly equal to 1, the maximum path difference is expressed by

$$\Delta r \approx r\,(1 - \cos\gamma\varphi_m) \approx \tfrac{1}{2} r\,(\gamma\varphi_m)^2 \tag{16·41}$$

The maximum phase delay is given by

$$\Delta\phi = k\,\Delta r \approx \tfrac{1}{2} k\, r\,(\gamma\varphi_m)^2 = \tfrac{1}{2}\cdot\gamma^2 r\,(k a^2)^{-1} \tag{16·42}$$

The pressure fluctuations are conveniently expressed by the coefficient of the amplitude variation which is defined by:

$$\begin{aligned} V_p^2 &= \frac{\overline{P^2} - (\overline{P})^2}{(\overline{P})^2} \approx \frac{\overline{(P_i + P_{sc}\cos\phi)^2}}{(\overline{P_i + P_{sc}\cos\phi})^2} - 1 \\ &= \frac{\overline{P_{sc}^2\cos^2\phi}}{P_i^2} = \beta\,\frac{I_{sc}}{I_i} \end{aligned} \tag{16·43}$$

where the coefficient β is the average of $\cos^2\phi$. Since $\phi = \pm\frac{\pi}{2} + \Delta\phi$ and the distribution of $\Delta\phi$ can be considered random,

$$\begin{aligned} \beta &= \frac{1}{2\Delta\phi}\left[\int_{-\pi/2}^{-\frac{\pi}{2}+\Delta\phi} \cos^2\phi\,d\phi + \int_{\pi/2}^{\frac{\pi}{2}+\Delta\phi} \cos^2\phi\,d\phi\right] \\ &= \tfrac{1}{2}\left[1 - (\sin 2\Delta\phi)/2\Delta\phi\right] \end{aligned} \tag{16·44}$$

For low frequencies or long distances, $\Delta\phi \gg 1$ and thus $\beta \approx \frac{1}{2}$. Therefore, Mintzer's interference scattering is obtained respectively

$$V_p^2 = k^2 \alpha a r \quad \text{or} \quad V_p^2 = \frac{1}{2}\sqrt{\pi}\, k^2 \alpha^2 a r \qquad (16.45a,b)$$

for the correlation function (16.35a) or (16.35b). For high frequencies or short distances ($\Delta\phi \lesssim 1$), $\beta \approx \frac{1}{3}(\Delta\phi)^2$. As the frequency increases, $\Delta\phi$ decreases but $|P_{sc}|$ increases and thus V_p^2 remains finite. In fact, if the distance r satisfies

$$r \approx k a^2 \qquad (16.46)$$

substitution of β from (16.44) and of $I_{\Delta C}$ from (16.38b) into (16.43) yields

$$V_p^2 = \frac{\sqrt{\pi}\, r^2 \gamma^4 k^2 a \alpha^2 r}{3\,(k a^2)^2 \cdot 4} = \frac{\sqrt{\pi}}{3}\,\frac{\gamma^4 \alpha^2 r^3}{4 a^3} \qquad (16.47)$$

The range $r \approx k a^2$ is identified as the focusing range which was discussed by Bergman (1946) in contrast with the interference range $r \gg k a^2$ discussed by Mintzer (1953). If $\gamma \approx 1.34$, equation (16.47) becomes identical with Bergman's equation

$$V_p^2 \approx \frac{1}{2}\,\alpha^2\, (r/a)^3 \qquad (16.48)$$

Whitmarsh and others (1957) and Urick and others (1958) discussed the data of simultaneous measurements of temperature

microstructure and sound scattering in various conditions of the ocean. Mean square temperature differences measured by two thermistors at the distances from 2 inch to 10 feet both in the horizontal and vertical directions show a good agreement with Kolmogoroff's 2/3 power law. The coefficient of amplitude variation V_P^2 was measured with 24 kc and 60 kc waves for the range of 50 to 1500 yards. The theoretical curves of V_P^2 were computed from equation (16·45b) and (16·47) using the sum of two Gaussian correlation functions with an assumption that the temperature fluctuations are cuased by two kinds of patches whose average scales are 35 and 150 inches. The observed relation bewteen V_P^2 and the range was reasonably explained by the theoretical curves thus obtained, although scattering in the data is large.

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